COLLEGE ALGEBRA

10TH EDITION

LIAL
HORNSBY
SCHNEIDER

4.1 Inverse Functions

Inverse Operations

Addition and subtraction are *inverse operations:* starting with a number *x,* adding 5, and subtracting 5 gives *x* back as the result. Similarly, some functions are *inverses* of each other. For example, the functions defined by

$$f(x) = 8x \text{ and } g(x) = \frac{1}{8}x$$

are inverses of each other with respect to function composition.

Inverse Operations

This means that if a value of *x* such as *x* = 12 is chosen, then

$$f(12) = 8 \cdot 12 = 96.$$

Calculating $g(96)$ gives

$$g(96) = \frac{1}{8} \cdot 96 = 12.$$

Inverse Operations

Thus $g(f(12)) = 12$. Also, $f(g(12)) = 12$. For these functions f *and* g, it can be shown that

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

for any value of *x.*

One-to-One Functions

Suppose we define the function

$$F = \{(-2,2), (-1,1), (0,0), (1,3), (2,5)\}.$$

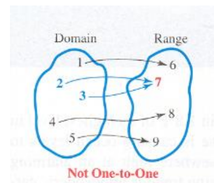
We can form another set of ordered pairs from *F* by interchanging the *x*- and *y*-values of each pair in *F*. We call this set *G*, so

$$G = \{(2,-2), (1,-1), (0,0), (3,1), (5,2)\}.$$

One-to-One Functions

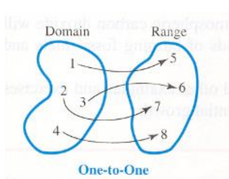
To show that these two sets are related, *G* is called the *inverse* of *F*. For a function f to have an inverse, f must be a *one-to-one function.* ***In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.***

One-to-One Functions



This function is not one-to-one because the *y*-value 7 corresponds to two *x*-values, 2 and 3. That is, the ordered pairs (2, 7) and (3, 7) both belong to the function.

One-to-One Functions



This function one-to-one.

One-to-One Function

A function f is a **one-to-one function** if, for elements *a* and *b* in the domain of f,

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

One-to-One Functions

Using the concept of the *contrapositive* from the study of logic, the last line in the preceding box is equivalent to

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

We use this statement to decide whether a function f is one-to-one in the next example.

Example 1 DECIDING WHETHER FUNCTIONS ARE ONE-TO-ONE

Decide whether each function is one-to-one.

a. $f(x) = -4x + 12$

Solution We must show that $f(a) = f(b)$ leads to the result $a = b$.

$$f(a) = f(b)$$

$$-4a + 12 = -4b + 12 \qquad f(x) = -4x + 12$$

$$-4a = -4b \qquad \text{Subtract 12.}$$

$$a = b \qquad \text{Divide by } -4.$$

By the definition, $f(x) = -4x + 12$ is one-to-one.

Example 1 DECIDING WHETHER FUNCTIONS ARE ONE-TO-ONE

Decide whether each function is one-to-one.

b. $f(x) = \sqrt{25 - x^2}$

Solution If we choose a = 3 and b = –3, then 3 ≠ –3, but

$$f(3) = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

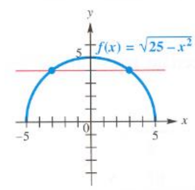
and $$f(-3) = \sqrt{25 - (-3)^2} = \sqrt{25 - 9} = 4.$$

Here, even though 3 ≠ –3, $f(3) = f(-3) = 4$. By the definition, f is *not* a one-to-one function.

Horizontal Line Test

As shown in Example 1(b), a way to show that a function is *not* one-to-one is to produce a pair of different numbers that lead to the same function value.

There is also a useful graphical test, the **horizontal line test,** that tells whether or not a function is one-to-one.



Horizontal Line Test

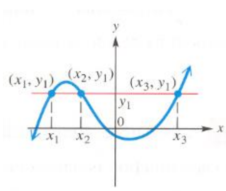
If any horizontal line intersects the graph of a function in no more than one point, then the function is one-to-one.

Note In Example 1(b), the graph of the function is a semicircle, as shown in Figure 3. Because there is at least one horizontal line that intersects the graph in more than one point, this function is not one-to-one.

Example 2 USING THE HORIZONTAL LINE TEST

Determine whether each graph is the graph of a one-to-one function.

a.



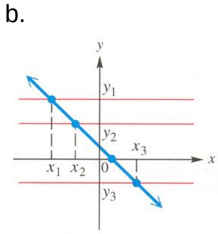
Solution

Each point where the horizontal line intersects the graph has the same value of y but a different value of x. Since more than one (here three) different values of x lead to the same value of y, the function is not one-to-one.

Example 2 USING THE HORIZONTAL LINE TEST

Determine whether each graph is the graph of a one-to-one function.

b.

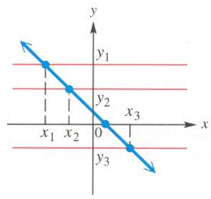


Solution

Since every horizontal line will intersect the graph at exactly one point, this function is one-to-one.

One-to-One Functions

Notice that the function graphed in Example 2(b) decreases on its entire domain. ***In general, a function that is either increasing or decreasing on its entire domain, such as $f(x) = -x$, $f(x) = x^3$, and $g(x) = \sqrt{x}$, must be one-to-one.***



Tests to Determine Whether a Function is One-to-One

1. Show that $f(a) = f(b)$ implies $a = b$. This means that f is one-to-one. (Example 1(a))

2. In a one-to-one function every y-value corresponds to no more than one x-value. To show that a function is not one-to-one, find at least two x-values that produce the same y-value. (Example 1(b))

Tests to Determine Whether a Function is One-to-One

3. Sketch the graph and use the horizontal line test. (Example 2)

4. If the function either increases or decreases on its entire domain, then it is one-to-one. A sketch is helpful here, too. (Example 2(b))

Inverse Functions

Certain pairs of one-to-one functions "undo" one another. For example, if

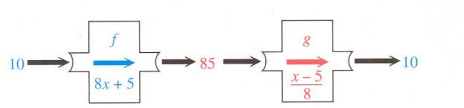
$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{x-5}{8},$$

then

$$f(10) = 8 \cdot 10 + 5 = 85 \quad \text{and} \quad g(85) = \frac{85-5}{8} = 10.$$

Inverse Functions

Starting with 10, we "applied" function f and then "applied" function g to the result, which returned the number 10.



Inverse Functions

As further examples, check that

$$f(3) = 29 \quad \text{and} \quad g(29) = 3,$$

$$f(-5) = -35 \quad \text{and} \quad g(-35) = -5,$$

$$g(2) = -\frac{3}{8} \quad \text{and} \quad g\left(-\frac{3}{8}\right) = 2,$$

Inverse Functions

In particular, for this pair of functions,

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2.$$

In fact, for any value of *x*,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

or $(f \circ g)(x) = x$ and $(g \circ f)(x) = x.$

Because of this property, g is called the *inverse* of f.

Inverse Function

Let f be a one-to-one function. Then g is the **inverse function** of f if

$(f \circ g)(x) = x$ for every *x* in the domain of *g*,

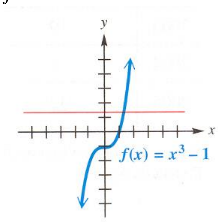
and

$(g \circ f)(x) = x$ for every *x* in the domain of f.

Example 3 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively. Is g the inverse function of f?

Solution The horizontal line test applied to the graph indicates that f is one-to-one, so the function does have an inverse. Since it is one-to-one, we now find $(f \circ g)(x)$ and $(g \circ f)(x)$.



Example 3 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively. Is g the inverse function of f?

Solution

$$(f \circ g)(x) = f(g(x)) = \left(\sqrt[3]{x+1}\right)^3 - 1$$

$$= x + 1 - 1$$

$$= x$$

Example 3 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively. Is g the inverse function of f?

Solution

$$(g \circ f)(x) = g(f(x)) = \sqrt[3]{(x^3 - 1) + 1}$$

$f(x) = x^3 - 1;$
$g(x) = \sqrt[3]{x+1}$

$$= \sqrt[3]{x^3}$$

$$= x$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

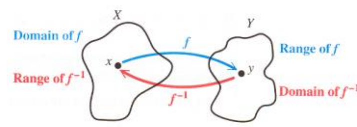
Special Notation

A special notation is used for inverse functions: If g is the inverse of a function f, then g is written as f^{-1} (read **"f-inverse"**). For $f(x) = x^3 - 1$, $f^{-1}(x) = \sqrt[3]{x+1}$

Caution ***Do not confuse the −1 in f^{-1} with a negative exponent.*** The symbol $f^{-1}(x)$ does not represent $\frac{1}{f(x)}$;it represents the inverse function of f.

Inverse Function

By the definition of inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.



Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

a. $F = \{(-2,1), (-1,0), (0,1), (1,2), (2,2)\}$

Solution Each x-value in F corresponds to just one y-value. However, the y-value 2 corresponds to two x-values, 1 and 2. Also, the y-value 1 corresponds to both −2 and 0. Because some y-values correspond to more than one x-value, F is not one-to-one and does not have an inverse.

Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

b. $G = \{(3,1), (0,2), (2,3), (4,0)\}$

Solution Every x-value in G corresponds to only one y-value, and every y-value corresponds to only one x-value, so G is a one-to-one function. The inverse function is found by interchanging the x- and y-values in each ordered pair.

$$G^{-1} = \{(1,3), (2,0), (3,2), (0,4)\}$$

Notice how the domain and range of G becomes the range and domain, respectively, of G^{-1}.

Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

c. If the Air Quality Index (AQI), an indicator of air quality, is between 101 and 150 on a particular day, then that day is classified as unhealthy for sensitive groups. The table shows the number of days in Illinois that were unhealthy for sensitive groups for selected years.

Year	Number of Unhealthy Days
2000	25
2001	40
2002	34
2003	19
2004	7
2005	32

Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of this function that is one-to-one.

c. Let f be the function defined in the table, with the years forming the domain and the numbers of unhealthy days forming the range.

Year	Number of Unhealthy Days
2000	25
2001	40
2002	34
2003	19
2004	7
2005	32

Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of this function that is one-to-one.

Solution Each x-value in f corresponds to only one y-value and each y-value corresponds to only one x-value, so f is a one-to-one function. The inverse function is found by interchanging the x- and y-values in the table.

Year	Number of Unhealthy Days
2000	25
2001	40
2002	34
2003	19
2004	7
2005	32

Example 4 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of this function that is one-to-one.

Solution

$$f^{-1}(x) = \{(25,2000),(40,2001),(34,2002),(19,2003),(7,2004),(32,2005)\}$$

Equations of Inverses

By definition, the inverse of a one-to-one function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows. (You may need to replace $f(x)$ with y first.)

Step 1 Interchange x and y.
Step 2 Solve for y.
Step 3 Replace y with $f^{-1}(x)$.

Example 5 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

a. $f(x) = 2x + 5$

Solution The graph of $y = 2x + 5$ is a non-horizontal line, so by the horizontal line test, f is a one-to-one function. To find the equation of the inverse, follow the steps in the preceding box, first replacing $f(x)$ with y.

Example 5 FINDING EQUATIONS OF INVERSES

Solution

$$y = 2x + 5 \qquad y = f(x)$$

$$x = 2y + 5 \qquad \text{Interchange } x \text{ and } y.$$

$$2y = x - 5 \qquad \text{Solve for } y.$$

$$y = \frac{x-5}{2}$$

$$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2} \qquad \text{Replace } y \text{ with } f^{-1}(x).$$

Example 5 **FINDING EQUATIONS OF INVERSES**

Solution

In the function, the value of *y* is found by starting with a value of *x,* multiplying by 2, and adding 5. The first form for the equation of the inverse has us *subtract* 5 and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable *x.*

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution The equation has a parabola opening up as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the x^2-term, there are many pairs of *x*-values that correspond to the same *y*-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution If we did not notice this, then following the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$

$$x = y^2 + 2 \quad \text{Interchange } x \text{ and } y.$$

$$x - 2 = y^2 \quad \text{Solve for } y.$$

$$\pm\sqrt{x-2} = y. \quad \text{Square root property}$$

Remember *both roots.*

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

b. $y = x^2 + 2$

Solution If we did not notice this, then following the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$

$$x = y^2 + 2$$

$$x - 2 = y^2$$

$$\pm\sqrt{x-2} = y.$$

Remember *both roots.*

The last step shows that there are two *y*-values for each choice of *x* greater than 2, so the given function is not one-to-one and cannot have an inverse.

Example 5 **FINDING EQUATIONS OF INVERSES**

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

c. $f(x) = (x-2)^3$

Solution Refer to **Sections 2.6 and 2.7** to see that translations of the graph of the cubing function are one-to-one.

$$f(x) = (x-2)^3$$

$$y = (x-2)^3 \quad \text{Replace } f(x) \text{ with } y.$$

$$x = (y-2)^3 \quad \text{Interchange } x \text{ and } y.$$

Example 5 **FINDING EQUATIONS OF INVERSES**

Solution

$$\sqrt[3]{x} = \sqrt[3]{(y-2)^3} \quad \text{Take the cube root on each side.}$$

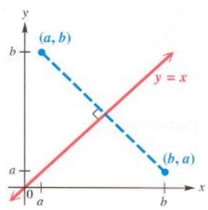
$$\sqrt[3]{x} = y - 2$$

$$\sqrt[3]{x} + 2 = y \quad \text{Solve for } y.$$

$$f^{-1}(x) = \sqrt[3]{x} + 2 \quad \text{Replace } y \text{ with } f^{-1}(x).$$

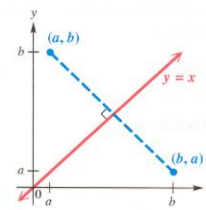
Inverse Function

One way to graph the inverse of a function f whose equation is known is to find some ordered pairs that are on the graph of f, interchange x and y to get ordered pairs that are on the graph of f^{-1}, plot those points, and sketch the graph of f^{-1} through the points.



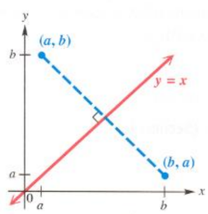
Inverse Function

A simpler way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.



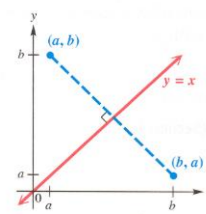
Inverse Function

For example, suppose the point (a, b) shown here is on the graph of a one-to-one function f.



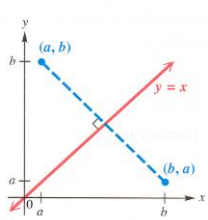
Inverse Function

Then the point (b, a) is on the graph of f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$.



Inverse Function

Thus, we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.



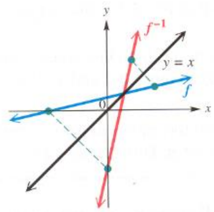
Example 6 **GRAPHING THE INVERSE**

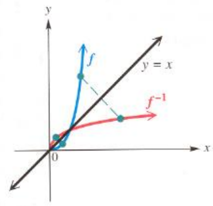
In each set of axes, the graph of a one-to-one function f is shown in blue. Graph f^{-1} in red.

Solution The graphs of two functions f *are* shown in blue. Their inverses are shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.

Example 6 **GRAPHING THE INVERSE**

Solution





Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

Let $f(x)=\sqrt{x+5}$. Find $f^{-1}(x)$.

Solution First, notice that the domain of f is restricted to the interval [–5, ∞). Function f is one-to-one because it is increasing on its entire domain and, thus, has an inverse function. Now we find the equation of the inverse.

Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

Solution

$f(x)=\sqrt{x+5}, \qquad x\geq -5$

$y=\sqrt{x+5}, \qquad x\geq -5 \qquad$ $y = f(x)$

$x=\sqrt{y+5}, \qquad y\geq -5 \qquad$ Interchange x and y.

$x^2=\left(\sqrt{y+5}\right)^2 \qquad$ Square both sides.

$y=x^2-5 \qquad$ Solve for y.

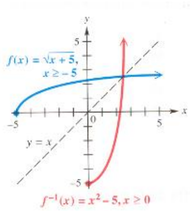
Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

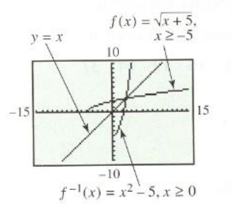
Solution However, we cannot define f^{-1} as $x^2 – 5$. The domain of f is [–5, ∞), and its range is [0, ∞).The range of f is the domain of f^{-1}, so f^{-1} must be defined as

$$f^{-1}(x)=x^2-5, \qquad x\geq 0.$$

Example 7 **FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN**

As a check, the range of f^{-1}, [–5, ∞), is the domain of f. Graphs of f and f^{-1} are shown. The line $y = x$ is included on the graphs to show that the graphs are mirror images with respect to this line.





Important Facts About Inverses

1. If f is one-to-one, then f^{-1} exists.
2. The domain of f is the range of f^{-1},and the range of f is the domain of f^{-1}.
3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}, so the graphs of f and f^{-1} are reflections of each other across the line $y = x$.
4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y. This gives $f^{-1}(x)$.

Application of Inverse Functions to Cryptography

A one-to-one function and its inverse can be used to make information secure. The function is used to encode a message, and its inverse is used to decode the coded message. In practice, complicated functions are used. We illustrate the process with the simple function defined by $f(x) = 3x + 1$. Each letter of the alphabet is assigned a numerical value according to its position in the alphabet, as follows.

Application of Inverse Functions to Cryptography

A	1	H	8	O	15	V	22
B	2	I	9	P	16	W	23
C	3	J	10	Q	17	X	24
D	4	K	11	R	18	Y	25
E	5	L	12	S	19	Z	26
F	6	M	13	T	20		
G	7	N	14	U	21		

Example 8 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function defined by $f(x) = 3x + 1$ and the preceding numerical values to encode and decode the message BE VERY CAREFUL.

Solution The message BE VERY CAREFUL would be encoded as

7 16 67 16 55 76 10 4 55 16 19 64 37

because B corresponds to 2 and

$$f(2) = 3(2) + 1 = 7,$$

Example 8 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function defined by $f(x) = 3x + 1$ and the preceding numerical values to encode and decode the message BE VERY CAREFUL.

Solution The message BE VERY CAREFUL would be encoded as

7 16 67 16 55 76 10 4 55 16 19 64 37

E corresponds to 5 and

$$f(5) = 3(5) + 1 = 16,$$

Example 8 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function defined by $f(x) = 3x + 1$ and the preceding numerical values to encode and decode the message BE VERY CAREFUL.

Solution The message BE VERY CAREFUL would be encoded as

7 16 67 16 55 76 10 4 55 16 19 64 37

and so on. Using the inverse $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$ to decode yields

$$f^{-1}(7) = \frac{1}{3}(7) + 1 = 2,$$

Example 8 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function defined by $f(x) = 3x + 1$ and the preceding numerical values to encode and decode the message BE VERY CAREFUL.

Solution The message BE VERY CAREFUL would be encoded as

7 16 67 16 55 76 10 4 55 16 19 64 37

$$f^{-1}(7) = \frac{1}{3}(7) + 1 = 2,$$

which corresponds to B,

Example 8 **USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE**

Use the one-to-one function defined by $f(x) = 3x + 1$ and the preceding numerical values to encode and decode the message BE VERY CAREFUL.

Solution The message BE VERY CAREFUL would be encoded as

7 16 67 16 55 76 10 4 55 16 19 64 37

$$f^{-1}(16) = \frac{1}{3}(16) + 1 = 5,$$

corresponds to E, and so on.