10TH EDITION

COLLEGE ALGEBRA

LIAL
HORNSBY
SCHNEIDER

PEARSON Education

4.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interest
- The Number *e* and Continuous Compounding
- Exponential Models and Curve Fitting

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Exponents and Properties

Recall the definition of a^r where r is a rational number: if $r = \frac{m}{n}$, then for appropriate values of m and n,

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

For example, $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$

Exponents and Properties

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3},$$

and

$$64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Exponents and Properties

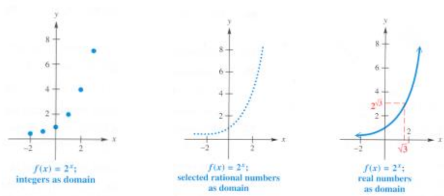
In this section we extend the definition of a^r to include all real (not just rational) values of the exponent *r.* For example, $2^{\sqrt{3}}$ might be evaluated by *approximating* the exponent $\sqrt{3}$ with the rational numbers 1.7, 1.73, 1.732, and so on.

Exponents and Properties

Since these decimals approach the value of $\sqrt{3}$ more and more closely, it seems reasonable that should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on.
Recall, for example, that
$2^{17/10} = \left(\sqrt[10]{2}\right)^{17}$.
This is exactly how $2^{\sqrt{3}}$ is defined (in a more advanced course).

Exponents and Properties

To show that this assumption is reasonable, see the graphs of the function $f(x) = 2^x$ with three different domains.



Exponents and Properties

Using this interpretation of real exponents, all rules and theorems for exponents are valid for all real number exponents, not just rational ones. In addition to the rules for exponents presented earlier, we use several new properties in this chapter. For example, if $y = 2^x$, then each real value of x leads to exactly one value of y, and therefore, defines a function. Furthermore,

Exponents and Properties

if $2^x = 2^4$, then $x = 4$,

and if $x = 4$, then $2^x = 2^4$,

Also, $4^2 < 4^3$ but $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$.

When $a > 1$, increasing the exponent on *"a"* leads to a larger number, but when $0 < a < 1$, increasing the exponent on *"a"* leads to a smaller number.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements are true.

(a) a^x is a unique real number for all real numbers x.

(b) $a^b = a^c$, if and only if $b = c$.

(c) If $a > 1$ and $m < n$, then $a^m < a^n$.

(d) If $0 < a < 1$ and $m < n$, then $a^m > a^n$.

Properties of Exponents

Properties (a) and (b) require $a > 0$ so that a^x is always defined. For example, $(-6)^x$ is not a real number if $x = ½$. This means that a^x will always be positive, since a must be positive. In property (a), a cannot equal 1 because $1^x = 1$ for every real number value of x, so each value of x leads to the same real number, 1. For property (b) to hold, a must not equal 1 since, for example, $1^4 = 1^5$, even though $4 \neq 5$.

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find the following.

a. $f(-1)$

Solution

$$f(-1) = 2^{-1} = \frac{1}{2}$$ Replace x with -1.

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find the following.

b. $f(3)$

Solution

$$f(3) = 2^3 = 8$$

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find the following.

c. $f\left(\frac{5}{2}\right)$

Solution

$$f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = 32^{1/2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find the following.

d. $f(4.92)$

Solution

$$f(4.92) = 2^{4.92} \approx 30.2738447$$ Use a calculator.

Exponential Function

If $a > 0$ and $a \neq 1$, then

$$f(x) = a^x$$

defines the **exponential function with base *a*.**

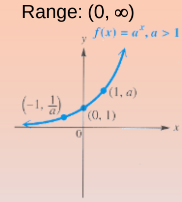
Note We do not allow 1 as the base for an exponential function. If $a = 1$, the function becomes the constant function defined by $f(x) = 1$, which is not an exponential function.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
−2	¼
−1	½
0	1
1	2
2	4
3	8



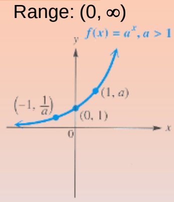
- $f(x) = a^x$, $a > 1$, is increasing and continuous on its entire domain, $(-\infty, \infty)$.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: (–∞, ∞) Range: (0, ∞)

For $f(x) = 2^x$:

x	$f(x)$
−2	¼
−1	½
0	1
1	2
2	4
3	8



- The x-axis is a horizontal asymptote as $x \to -\infty$.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: (–∞, ∞) Range: (0, ∞)

For $f(x) = 2^x$:

x	$f(x)$
−2	¼
−1	½
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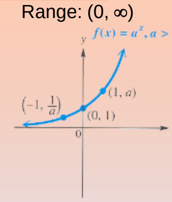


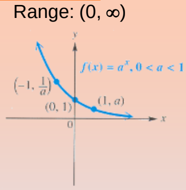
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, (0,1), and (1,$a$).

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: (–∞, ∞) Range: (0, ∞)

For $f(x) = (½)^x$:

x	$f(x)$
−3	8
−2	4
−1	2
0	1
1	½
2	¼



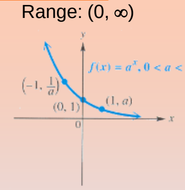
- $f(x) = a^x$, 0 < a < 1, is decreasing and continuous on its entire domain, (–∞, ∞) .

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: (–∞, ∞) Range: (0, ∞)

For $f(x) = (½)^x$:

x	$f(x)$
−3	8
−2	4
−1	2
0	1
1	½
2	¼



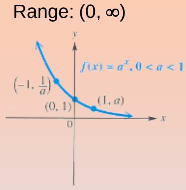
- The x-axis is a horizontal asymptote as $x \to \infty$.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: (–∞, ∞) Range: (0, ∞)

For $f(x) = (½)^x$:

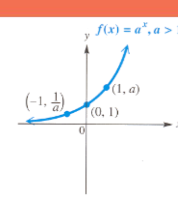
x	$f(x)$
−3	8
−2	4
−1	2
0	1
1	½
2	¼

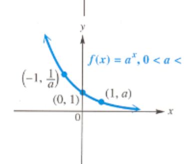


- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, (0,1), and (1,$a$).

Exponential Function

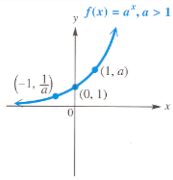
Starting with $f(x) = 2^x$ and replacing x with $-x$ gives $f(-x) = 2^{-x} = (2^{-1})^x = (½)^x$. For this reason, their graphs are reflections of each other across the y-axis.



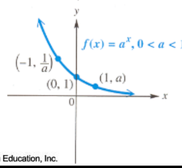


Exponential Function

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar.

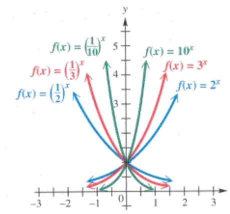


When $0 < a < 1$, the graph decreases in a manner similar to this graph



Exponential Function

The graphs of several typical exponential functions illustrate these facts.



$f(x) = a^x$
Domain: $(-\infty, \infty)$; Range: $(0, \infty)$
When $a > 1$, the function is increasing.
When $0 < a < 1$, the function is decreasing.
In every case, the x-axis is a horizontal asymptote.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1,a)$ and are on the graph.
2. If $a > 1$, then f is an increasing function; if $0 < a < 1$, then f is a decreasing function.
3. The x-axis is a horizontal asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

Example 4 USING A PROPERTY OF EXPONENTS TO SOLVE AN EQUATION

Solve $\left(\frac{1}{3}\right)^x = 81.$

Solution ***Write each side of the equation using a common base.***

$$\left(\frac{1}{3}\right)^x = 81$$

$$\left(3^{-1}\right)^x = 81 \qquad \text{Definition of negative exponent.}$$

Example 4 USING A PROPERTY OF EXPONENTS TO SOLVE AN EQUATION

Solve $\left(\frac{1}{3}\right)^x = 81.$

Solution ***Write each side of the equation using a common base.***

$$3^{-x} = 81 \qquad (a^m)^n = a^{mn}$$

$$3^{-x} = 3^4 \qquad \text{Write 81 as a power of 3.}$$

$$-x = 4 \qquad \text{Set exponents equal.}$$

$$x = -4 \qquad \text{Multiply by } -1.$$

The solution set of the original equation is $\{-4\}$.

Example 5 USING A PROPERTY OF EXPONENTS TO SOLVE AN EQUATION

Solve $2^{x+4} = 8^{x-6}$.

Solution Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6} \qquad \text{Write 8 as a power of 2.}$$

$$2^{x+4} = 2^{3x-18} \qquad (a^m)^n = a^{mn}$$

$$x + 4 = 3x - 18 \qquad \text{Set exponents equal .}$$

$$-2x = -22 \qquad \text{Subtract } 3x \text{ and } 4.$$

$$x = 11 \qquad \text{Divide by } -2.$$

Check by substituting 11 for x in the original equation. The solution set is $\{11\}$.

Example 6 **USING A PROPERTY OF EXPONENTS TO SOLVE AN EQUATION**

Solve $b^{4/3}=81.$

Solution Begin by writing $b^{4/3}$ as $\left(\sqrt[3]{b}\right)^4.$

$\left(\sqrt[3]{b}\right)^4=81$ Radical notation for $a^{m/n}$

$\sqrt[3]{b}=\pm 3$ Take fourth root on both sides.

$b=\pm 27$ Cube both sides.

Check *both* solutions in the original equation. Both check, so the solution set is {–27, 27}

Compound Interest

The formula for **compound interest** (interest paid on both principal and interest) is an important application of exponential functions. Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose t = 1 yr. Then at the end of the year the amount has grown to

$$P+Pr=P(1+r),$$

Compound Interest

the original principal plus interest. If this balance earns interest at the same interest rate for another year, the balance at the end of that year will be

$$[P(1+r)]+[P(1+r)]r=[P(1+r)](1+r) \quad \text{Factor.}$$

$$=P(1+r)^2.$$

Compound Interest

After the third year, this will grow to

$$[P(1+r)^2]+[P(1+r)^2]r=[P(1+r)^2](1+r) \quad \text{Factor.}$$

$$=P(1+r)^3.$$

Continuing in this way produces a formula for interest compounded annually.

$$A=P(1+r)^t$$

Compound Interest

If P dollars are deposited in an account paying an annual rate of interest r compounded (paid) n times per year, then after t years the account will contain A dollars, where

$$A=P\left(1+\frac{r}{n}\right)^{tn}.$$

Example 7 **USING THE COMPOUND INTEREST FORMULA**

Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

a. Find the amount in the account after 10 yr with no withdrawals.

Solution

$$A=P\left(1+\frac{r}{n}\right)^{tn} \quad \text{Compound interest formula}$$

$$A=1000\left(1+\frac{.04}{4}\right)^{10(4)} \quad \text{Let } P = 1000,\ r = .04,\ n = 4, \text{ and } t = 10.$$

Example 7 **USING THE COMPOUND INTEREST FORMULA**

Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

a. Find the amount in the account after 10 yr with no withdrawals.

Solution

$$A = 1000(1 + .01)^{40}$$

$$A = 1488.86$$ Round to the nearest cent.

Thus, $1488.86 is in the account after 10 yr.

Example 7 **USING THE COMPOUND INTEREST FORMULA**

Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

b. How much interest is earned over the 10-yr period?

Solution The interest earned for that period is

$$\$1488.86 - \$1000 = \$488.86$$

Example 8 **FINDING A PRESENT VALUE**

Becky Anderson must pay a lump sum of $6000 in 5 yr.

a. What amount deposited today at 3.1% compounded annually will grow to $6000 in 5 yr?

Solution

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$ Compound interest formula

$$6000 = P\left(1 + \frac{.031}{1}\right)^{5(1)}$$ Let A = 6000, r = .031, n = 1, and t =5.

Example 8 **FINDING A PRESENT VALUE**

Becky Anderson must pay a lump sum of $6000 in 5 yr.

a. What amount deposited today at 3.1% compounded annually will grow to $6000 in 5 yr?

Solution

$$6000 = P(1.031)^5$$ Simplify.

$$P \approx 5150.60$$ Use a calculator.

If Becky leaves $5150.60 for 5 yr in an account paying 3.1% compounded annually, she will have $6000 when she needs it. We say that $5150.60 is the present value of $6000 if interest of 3.1% is compounded annually for 5 yr.

Example 8 **FINDING A PRESENT VALUE**

Becky Anderson must pay a lump sum of $6000 in 5 yr.

b. If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yr?

Solution

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$6000 = 5000(1 + r)^5$$ Let A = 6000, P = 5000, n = 1, and t =5.

Example 8 **FINDING A PRESENT VALUE**

Becky Anderson must pay a lump sum of $6000 in 5 yr.

b. If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yr?

Solution

$$\frac{6}{5} = (1 + r)^5$$ Divide by 5000.

$$\left(\frac{6}{5}\right)^{1/5} = 1 + r$$ Take the fifth root on both sides.

Example 8 **FINDING A PRESENT VALUE**

Becky Anderson must pay a lump sum of $6000 in 5 yr.

b. If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yr?

Solution

$$\left(\frac{6}{5}\right)^{1/5} - 1 = r \qquad \text{Subtract 1.}$$

$$r \approx .0371 \qquad \text{Use a calculator.}$$

An interest rate of 3.71% will produce enough interest to increase the $5000 to $6000 by the end of 5 yr.

Continuous Compounding

The more often interest is compounded within a given time period, the more interest will be earned. Surprisingly, however, there is a limit on the amount of interest, no matter how often it is compounded.

Continuous Compounding

Suppose that $1 is invested at 100% interest per year, compounded n times per year. Then the interest rate (in decimal form) is 1.00 and the interest rate per period is $\frac{1}{n}$. According to the formula (with $P = 1$), the compound amount at the end of 1 yr will be

$$A = \left(1 + \frac{1}{n}\right)^n.$$

Continuous Compounding

A calculator gives the results for various values of n. The table suggests that as n increases, the value of $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to some fixed number. This is indeed the case. This fixed number is called e. ***(Note that in mathematics, e is a real number and not a variable.)***

n	$\left(1+\frac{1}{n}\right)^n$ (rounded)
1	2
2	2.25
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
1,000,000	2.71828

Value of e

To nine decimal places,

$$e \approx 2.718281828.$$

Continuous Compounding

If P dollars are deposited at a rate of interest r compounded continuously for t years, the compound amount in dollars on deposit is

$$A = Pe^{rt}.$$

Example 9 SOLVING A CONTINUOUS COMPOUNDING PROBLEM

Suppose $5000 is deposited in an account paying 3% interest compounded continuously for 5 yr. Find the total amount on deposit at the end of 5 yr.

Solution

$A = Pe^{rt}$ — Continuous compounding formula.

$= 5000e^{.03(5)}$ — Let P = 5000, t = 5, and r = .03.

$= 5000e^{.15}$

≈ 5809.17 — Use a calculator.

or $5809.17. Check that daily compounding would have produced a compound amount about 3¢ less.

Example 10 COMPARE INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

In Example 7, we found that $1000 invested at 4% compounded quarterly for 10 yr grew to $1488.86. Compare this same investment compounded annually, semiannually, monthly, daily, and continuously.

Solution

Substituting into the compound interest formula and the formula for continuous compounding gives the following results for amounts of $1 and $1000.

Example 10 COMPARE INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

Compounded	$1	$1000
Annually	$(1+.04)^{10} \approx 1.48024$	$1480.24
Semiannually	$\left(1+\frac{.04}{2}\right)^{10(2)} \approx 1.48595$	$1485.95
Quarterly	$\left(1+\frac{.04}{4}\right)^{10(4)} \approx 1.48886$	$1488.86
Monthly	$\left(1+\frac{.04}{12}\right)^{10(12)} \approx 1.49083$	$1490.83
Daily	$\left(1+\frac{.04}{365}\right)^{10(365)} \approx 1.49179$	$1491.79
Continuously	$e^{10(.04)} \approx 1.49182$	$1491.82

Example 10 COMPARE INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

Compounding semiannually rather than annually increases the value of the account after 10 yr by $5.71. Quarterly compounding grows to $2.91 more than semiannual compounding after 10 yr. Daily compounding yields only $.96 more than monthly compounding. Each increase in the frequency of compounding earns less and less additional interest, until going from daily to continuous compounding increases the value by only .03.