10[TH] EDITION

COLLEGE ALGEBRA

LIAL
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SCHNEIDER

4.3 Logarithmic Functions

Logarithms

The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of *a*, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one, and thus have inverse functions.

Logarithms

The equation defining the inverse of a function is found by interchanging *x* and *y* in the equation that defines the function. Starting with $y = a^x$ and interchanging *x* and *y* yields

$$x = a^y.$$

Logarithms

$$x = a^y$$

Here *y* is the exponent to which *a* must be raised in order to obtain *x*. We call this exponent a **logarithm, *symbolized by* "log."** The expression $\log_a x$ represents the logarithm in this discussion. The number *a* is called the **base** of the logarithm, and *x* is called the **argument** of the expression. It is read **"logarithm with base *a* of *x*,"** or **"logarithm of *x* with base *a*."**

Logarithm

For all real numbers *y* and all positive numbers *a* and *x*, where $a \neq 1$,

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

A logarithm is an exponent. The expression $\log_a x$ represents the exponent to which the base "a" must be raised in order to obtain x.

Logarithms

Exponent ↓

Logarithmic form: $y = \log_a x$

↑ Base

Exponent ↓

Exponential form: $a^y = x$

↑ Base

Logarithms

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

Example 1 SOLVING LOGARITHMIC EQUATIONS

Solve

a. $\log_x \frac{8}{27} = 3$

Solution $\log_x \frac{8}{27} = 3$

$x^3 = \frac{8}{27}$ Write in exponential form.

$x^3 = \left(\frac{2}{3}\right)^3$ $\frac{8}{27} = \left(\frac{2}{3}\right)^3$

$x = \frac{2}{3}$ Take cube roots

Example 1 SOLVING LOGARITHMIC EQUATIONS

Check: $\log_x \frac{8}{27} = 3$ Original equation

$\log_{2/3} \frac{8}{27} = 3$? Let x = ⅔.

$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$? Write in exponential form

$\frac{8}{27} = \frac{8}{27}$ True

The solution set is $\left\{\frac{2}{3}\right\}$.

Example 1 SOLVING LOGARITHMIC EQUATIONS

Solve

b. $\log_4 x = \frac{5}{2}$

Solution $\log_4 x = \frac{5}{2}$

$4^{5/2} = x$ Write in exponential form.

$(4^{1/2})^5 = x$ $a^{mn} = (a^m)^n$

$2^5 = x$ $4^{1/2} = (2^2)^{1/2} = 2$

$32 = x$

The solution set is {32}.

Example 1 SOLVING LOGARITHMIC EQUATIONS

Solve

c. $\log_{49} \sqrt[3]{7} = x$

Solution $49^x = \sqrt[3]{7}$ Write in exponential form.

$(7^2)^x = 7^{1/3}$ Write with the same base.

$7^{2x} = 7^{1/3}$ Power rule for exponents.

$2x = \frac{1}{3}$ Set exponents equal.

$x = \frac{1}{6}$ Divide by 2.

The solution set is $\left\{\frac{1}{6}\right\}$.

Logarithmic Function

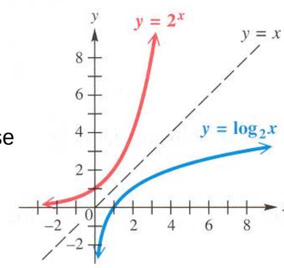
If $a > 0$, $a \neq 1$, and $x > 0$, then

$$f(x) = \log_a x$$

defines the **logarithmic function with base *a*.**

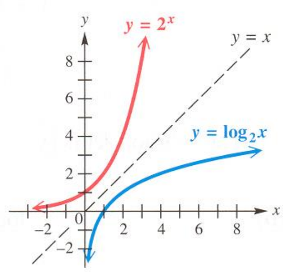
Logarithmic Function

Exponential and logarithmic functions are inverses of each other. The graph of $y = 2^x$ is shown in red. The graph of its inverse is found by reflecting the graph across the line $y = x$.



Logarithmic Function

The graph of the inverse function, defined by $y = \log_2 x$, shown in blue, has the *y*-axis as a vertical asymptote.



Logarithmic Function

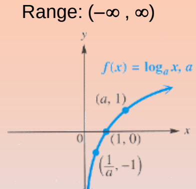
Since the domain of an exponential function is the set of all real numbers, the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers, so ***logarithms can be found for positive numbers only.***

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
¼	−2
½	−1
1	0
2	1
4	2
8	3



- $f(x) = \log_a x$, $a > 1$, is increasing and continuous on its entire domain, $(0, \infty)$.

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

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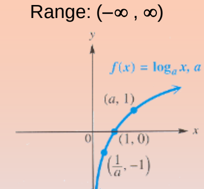

- The *y*-axis is a vertical asymptote as $x \to 0$ from the right.

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

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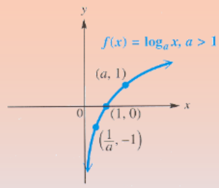


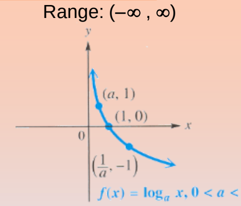
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1,0)$, and $(a,1)$.

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

x	$f(x)$
¼	2
½	1
1	0
2	−1
4	−2
8	−3



- $f(x) = \log_a x$, $0 < a < 1$, is decreasing and continuous on its entire domain, $(0, \infty)$.

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

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¼	2
½	1
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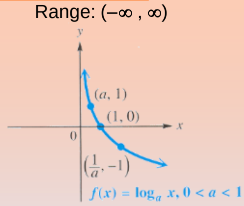

- The y-axis is a vertical asymptote as $x \to 0$ from the right.

LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

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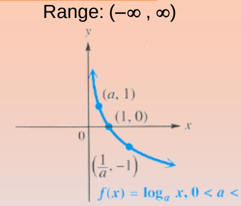

- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1,0)$, and $(a,1)$.

Characteristics of the Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1,0)$, and $(a,1)$ are on the graph.
2. If $a > 1$, then f is an increasing function; if $0 < a < 1$, then f is a decreasing function.
3. The y-axis is a vertical asymptote.
4. The domain is $(0,\infty)$, and the range is $(-\infty, \infty)$.

▶ **Caution** If you write a logarithmic function in exponential form, choosing y-values to calculate x-values, ***be careful to write the values in the ordered pairs in the correct order.***

Properties of Logarithms

Since a logarithmic statement can be written as an exponential statement, it is not surprising that the properties of logarithms are based on the properties of exponents. The properties of logarithms allow us to change the form of logarithmic statements so that products can be converted to sums, quotients can be converted to differences, and powers can be converted to products.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r:

Property	Description
Product Property $\log_a xy = \log_a x + \log_a y$	The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r:

Property	Description
Quotient Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r:

Property	Description
Power Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of a number raised to a power is equal to the exponent multiplied by the logarithm of the number.

Properties of Logarithms

Two additional properties of logarithms follow directly from the definition of $\log_a x$ since $a^0 = 1$ and $a^1 = a$.

$$\log_a 1 = 0 \text{ and } \log_a a = 1$$

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

a. $\log_6(7 \cdot 9)$

Solution

$$\log_6(7 \cdot 9) = \log_6 7 + \log_6 9$$ Product property

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

b. $\log_9 \frac{15}{7}$

Solution

$$\log_9 \frac{15}{7} = \log_9 15 - \log_9 7$$ Quotient property

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

c. $\log_5 \sqrt{8}$

Solution

$$\log_5 \sqrt{8} = \log_5 (8^{1/2}) = \frac{1}{2}\log_5 8$$ Power property

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

d. $\log_a \frac{mnq}{p^2 t^4}$

Use parentheses to avoid errors.

Solution

$$\log_a \frac{mnq}{p^2 t^4} = \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a t^4)$$

$$= \log_a m + \log_a n + \log_a q - (2\log_a p + 4\log_a t)$$

$$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a t$$

Be careful with signs.

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

e. $\log_a \sqrt[3]{m^2}$

Solution

$$\log_a \sqrt[3]{m^2} = \log_a m^{2/3} = \frac{2}{3}\log_a m$$ Power property

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

f.

$$\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} = \log_b \left(\frac{x^3 y^5}{z^m}\right)^{1/n}$$ $\sqrt[n]{a} = a^{1/n}$

$$= \frac{1}{n}\log_b \frac{x^3 y^5}{z^m}$$ Power property

$$= \frac{1}{n}(\log_b x^3 + \log_b y^5 - \log_b z^m)$$ Product and quotient properties

Example 4 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with a ≠ 1 and b ≠ 1.

f. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} = \log_b \left(\frac{x^3 y^5}{z^m}\right)^{1/n}$

Solution

$$= \frac{1}{n}(3\log_b x + 5\log_b y - m\log_b z)$$ Power property

$$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$$ Distributive property

Example 5 **USING THE PROPERTIES OF LOGARITHMS**

Write the expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$and $b \neq 1$.

a. $\log_3(x+2)+\log_3 x-\log_3 2$

Solution

$$\log_3(x+2)+\log_3 x-\log_3 2=\log_3\frac{(x+2)x}{2}$$

Product and quotient properties

Example 5 **USING THE PROPERTIES OF LOGARITHMS**

Write the expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$and $b \neq 1$.

b. $2\log_a m-3\log_a n$

Solution

$$2\log_a m-3\log_a n=\log_a m^2-\log_a n^3$$

Power property

$$=\log_a\frac{m^2}{n^3}$$

Quotient property

Example 5 **USING THE PROPERTIES OF LOGARITHMS**

Write the expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$and $b \neq 1$.

c. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2 n$

Solution

$$\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2 n$$

$$=\log_b m^{1/2}+\log_b(2n)^{3/2}-\log_b m^2 n$$ Power properties

Example 5 **USING THE PROPERTIES OF LOGARITHMS**

Write the expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$and $b \neq 1$.

c. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2 n$

Solution

$$=\log_b\frac{m^{1/2}(2n)^{3/2}}{m^2 n}$$ Product and quotient properties

$$=\log_b\frac{2^{3/2}n^{1/2}}{m^{3/2}}$$ Rules for exponents

Example 5 **USING THE PROPERTIES OF LOGARITHMS**

Write the expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$and $b \neq 1$.

c. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2 n$

Solution

$$=\log_b\left(\frac{2^3 n}{m^3}\right)^{1/2}$$ Rules for exponents

$$=\log_b\sqrt{\frac{8n}{m^3}}$$ Definition of $a^{1/n}$

Caution ***There is no property of logarithms to rewrite a logarithm of a sum or difference.*** That is why, in Example 5(a), $\log_3(x + 2)$ was not written as $\log_3 x + \log_3 2$. Remember, $\log_3 x + \log_3 2 = \log_3(x \cdot 2)$.

The distributive property does not apply in a situation like this because $\log_3 (x + y)$ is one term; "log" is a function name, not a factor.

Example 6 USING THE PROPERTIES OF LOGARITHMS WITH NUMERICAL VALUES

Assume that $\log_{10} 2 = .3010$. Find each logarithm.

a. $\log_{10} 4$

Solution

$$\log_{10} 4 = \log_{10} 2^2 = 2 \log_{10} 2 = 2(.3010) = .6020$$

b. $\log_{10} 5$

$$\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - .3010 = .6990$$

Theorem on Inverses

For $a > 0$, $a \neq 1$:

$$a^{\log_a x} = x \text{ and } \log_a a^x = x.$$

Theorem on Inverses

By the results of this theorem,

$$7^{\log_7 10} = 10, \quad \log_5 5^3 = 3, \quad \text{and} \quad \log_r r^{k+1} = k + 1.$$

The second statement in the theorem will be useful in **Sections 4.5 and 4.6** when we solve other logarithmic and exponential equations.