10TH EDITION

COLLEGE ALGEBRA

LIAL
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SCHNEIDER

4.4 Evaluating Logarithms and the Change-of-Base Theorem

- Common Logarithms
- Applications and Modeling with Common Logarithms
- Natural Logarithms
- Applications and Modeling with Natural Logarithms
- Logarithms with Other Bases

Common Logarithm

For all positive numbers *x*,

$$\log x = \log_{10} x.$$

Common Logarithms

A calculator with a log key can be used to find the base ten logarithm of any positive number. Consult your owner's manual for the keystrokes needed to find common logarithms.

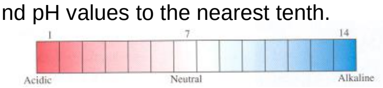
Note ***Base a, a > 1, logarithms of numbers between*** **0** ***and*** **1** ***are always negative,*** as suggested by the graphs in **Section 4.3.**

Applications and Modeling

In chemistry, the **pH** of a solution is defined as

$$\text{pH} = -\log[H_3O^+],$$

where $[H_3O^+]$ is the hydronium ion concentration in moles per liter. The pH value is a measure of the acidity or alkalinity of a solution. Pure water has pH 7.0, substances with pH values greater than 7.0 are alkaline, and substances with pH values less than 7.0 are acidic. It is customary to round pH values to the nearest tenth.



Example 1 **FINDING pH**

a. Find the pH of a solution with $[H_3O^+] = 2.5 \times 10^{-4}$.

Solution

$$pH = -\log[H_3O^+]$$

$$= -\log[2.5 \times 10^{-4}]$$ Substitute

$$= -(\log 2.5 + \log 10^{-4})$$ Product property

$$= -(.3979 - 4)$$ $\log 10^{-4} = -4$

$$= -.3979 + 4$$ Distributive property

$$pH \approx 3.6$$

Example 1 **FINDING pH**

b. Find the hydronium ion concentration of a solution with pH = 7.1.

Solution

$$pH = -\log[H_3O^+]$$

$$7.1 = -\log[H_3O^+]$$ Substitute

$$-7.1 = \log[H_3O^+]$$ Multiply by −1.

$$[H_3O^+] = 10^{-7.1}$$ Write in exponential form.

$$[H_3O^+] \approx 7.9 \times 10^{-8}$$ Evaluate $10^{-7.1}$ with a calculator.

Note In the fourth line of the solution in Example 1(a), we use the equality symbol, =, rather than the approximate equality symbol, ≈, when replacing log 2.5 with .3979. This is often done for convenience, despite the fact that most logarithms used in applications are indeed approximations.

Example 2 **USING pH IN AN APPLICATION**

Wetlands are classified as *bogs, fens, marshes, and swamps.* These classifications are based on pH values. A pH value between 6.0 and 7.5, such as that of Summerby Swamp in Michigan's Hiawatha National Forest, indicates that the wetland is a "rich fen." When the pH is between 4.0 and 6.0, it is a "poor fen," and if the pH falls to 3.0 or less, the wetland is a "bog." Suppose that the hydronium ion concentration of a sample of water from a wetland is 6.3×10^{-5}. How would this wetland be classified?

Example 2 **USING pH IN AN APPLICATION**

Solution

$$pH = -\log[H_3O^+]$$ Definition of pH

$$= -\log(6.3 \times 10^{-5})$$ Substitute

$$= -(\log 6.3 + \log 10^{-5})$$ Product property

$$= -\log 6.3 - (-5)$$ Distributive property

$$= -\log 6.3 + 5$$

$$pH \approx 4.2$$ Since the pH is between 4.0 and 6.0, the wetland is a poor fen.

Example 3 **MEASURING THE LOUDNESS OF SOUND**

The loudness of sounds is measured in a unit called a **decibel.** To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the **threshold sound.** If a particular sound has intensity $\boldsymbol{I}$, then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}.$$

Find the decibel rating of a sound with intensity $10{,}000I_0$.

Example 3 **MEASURING THE LOUDNESS OF SOUND**

Solution

$$d = 10\log\frac{10{,}000I_0}{I_0}$$ Let $I = 10{,}000I_0$.

$$d = 10\log 10{,}000$$

$$= 10(4)$$ $\log 10{,}000 = \log 10^4 = 4$

$$= 40$$

The sound has a decibel rating of 40.

Natural Logarithms

In **Section 4.2,** we introduced the irrational number *e*. In most practical applications of logarithms, *e* is used as base. Logarithms with base *e* are called **natural logarithms,** since they occur in the life sciences and economics in natural situations that involve growth and decay. The base *e* logarithm of *x* is written **ln *x*** (read ***"el-en x"***). ***The expression ln x represents the exponent to which e must be raised in order to obtain x.***

Natural Logarithm

For all positive numbers *x*,

$$\ln x = \log_e x.$$

Example 4 **MEASURING THE AGE OF ROCKS**

Geologists sometimes measure the age of rocks by using "atomic clocks." By measuring the amounts of potassium 40 and argon 40 in a rock, the age *t* of the specimen in years is found with the formula

$$t = (1.26\times 10^9)\frac{\ln\left(1+8.33\left(\frac{A}{K}\right)\right)}{\ln 2},$$

where *A and K* are the numbers of atoms of argon 40 and potassium 40, respectively, in the specimen.

Example 4 **MEASURING THE AGE OF ROCKS**

a. How old is a rock in which $A = 0$ and $K > 0$?

Solution

If $A = 0$, $\frac{A}{K} = 0$ and the equation becomes

$$t = (1.26\times 10^9)\frac{\ln\left(1+8.33\left(\frac{A}{K}\right)\right)}{\ln 2} = (1.26\times 10^9)\frac{\ln 1}{\ln 2}$$

$$= (1.26\times 10^9)(0) = 0.$$

The rock is new (0 yr old).

Example 4 **MEASURING THE AGE OF ROCKS**

b. The ratio $\frac{A}{K}$ for a sample of granite from New Hampshire is .212. How old is the sample?

Solution

Since $\frac{A}{K} = .212$, we have

$$t = (1.26\times 10^9)\frac{\ln(1+8.33(.212))}{\ln 2} \approx 1.85\times 10^9.$$

The granite is about 1.85 billion yr old.

Example 5 MODELING GLOBAL TEMPERATURE INCREASE

Carbon dioxide in the atmosphere traps heat from the sun. The additional solar radiation trapped by carbon dioxide is called **radiative forcing.** It is measured in watts per square meter (w/m^2). In 1896 the Swedish scientist Svante Arrhenius modeled radiative forcing R caused by additional atmospheric carbon dioxide using the logarithmic equation

$$R = k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide, C is the current carbon dioxide level, and k is a constant. Arrhenius determined that $10 \le k \le 16$ when $C = 2C_0$.

Example 5 MODELING GLOBAL TEMPERATURE INCREASE

a. Let $C = 2C_0$. Is the relationship between R and k linear or logarithmic?

Solution

If $C = 2C_0$, then $\frac{C}{C_0} = 2$, so $R = k \ln 2$ is a linear relation, because ln 2 is a constant.

Example 5 MODELING GLOBAL TEMPERATURE INCREASE

b. The average global temperature increase T (in °F) is given by $T(R) = 1.03R$. Write T as a function of k.

Solution

$$T(R) = 1.03R$$

$$T(k) = 1.03k \ln \frac{C}{C_0}$$

Use the given expression for R.

Logarithms and Other Bases

We can use a calculator to find the values of either natural logarithms (base e) or common logarithms (base 10). However, sometimes we must use logarithms with other bases. The following theorem can be used to convert logarithms from one base to another.

Change-of-Base Theorem

For any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Example 6 USING THE CHANGE-OF-BASE THEOREM

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm.

a. $\log_5 17$

Solution

We will arbitrarily use natural logarithms.

There is no need to actually write this step.

$$\log_5 17 = \frac{\ln 17}{\ln 5} \approx \frac{2.8332}{1.6094} \approx 1.7604$$

Example 6 **USING THE CHANGE-OF-BASE THEOREM**

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm.

b. $\log_2 .1$

Solution

Here we use common logarithms.

$$\log_2 .1 = \frac{\log .1}{\log 2} \approx -3.3219$$

Note In Example 6, logarithms evaluated in the intermediate steps, such as ln 17 and ln 5, were shown to four decimal places. However, the final answers were obtained *without* rounding these intermediate values, using all the digits obtained with the calculator. ***In general, it is best to wait until the final step to round off the answer; otherwise, a build-up of round-off errors may cause the final answer to have an incorrect final decimal place digit.***

Example 7 **MODELING DIVERSITY OF SPECIES**

One measure of the diversity of the species in an ecological community is modeled by the formula

$$H = -[P_1 \log_2 P_1 + P_2 \log_2 P_2 + \cdots + P_n \log_2 P_n],$$

where $P_1, P_2, \ldots, P_n$ are the proportions of a sample that belong to each of n species found in the sample.

Find the measure of diversity in a community with two species where there are 90 of one species and 10 of the other.

Example 7 **MODELING DIVERSITY OF SPECIES**

Solution

Since there are 100 members in the community,

$$P_1 = \frac{90}{100} = .9 \text{ and } P_2 = \frac{10}{100} = .1, \text{ so}$$

$$H = -[.9 \log_2 .9 + .1 \log_2 .1].$$

Now we find $\log_2 .9$.

$$\log_2 .9 = \frac{\log .9}{\log 2} \approx -.152$$

Example 7 **MODELING DIVERSITY OF SPECIES**

Solution

Therefore,

$$H = -[.9 \log_2 .9 + .1 \log_2 .1]$$

Found in Example 6b.

$$\approx -[.9(-.152) + .1(-3.32)] \approx .469$$

Verify that $H \approx .971$ if there are 60 of one species and 40 of the other. As the proportions of n species get closer to $\frac{1}{n}$ each, the measure of diversity increases to a maximum of $\log_2 n$.