COLLEGE ALGEBRA

10TH EDITION

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4.5 Exponential and Logarithmic Equations

Exponential Equations
Logarithmic Equations
Applications and Modeling

Property of Logarithms

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then

$x = y$ if and only if $\log_a x = \log_a y$.

Example 1 SOLVING AN EXPONENTIAL EQUATION

Solve $7^x = 12$. Give the solution to the nearest thousandth.

Solution

The properties of exponents given in **Section 4.2** cannot be used to solve this equation, so we apply the preceding property of logarithms. While any appropriate base *b* can be used, the best practical base is base 10 or base *e*. We choose base *e* (natural) logarithms here.

Example 1 SOLVING AN EXPONENTIAL EQUATION

Solve $7^x = 12$. Give the solution to the nearest thousandth.

Solution

$$7^x = 12$$

$$\ln 7^x = \ln 12 \quad \text{Property of logarithms}$$

$$x \ln 7 = \ln 12 \quad \text{Power of logarithms}$$

$$x = \frac{\ln 12}{\ln 7} \quad \text{Divide by ln 7.}$$

$$x \approx 1.277 \quad \text{Use a calculator.}$$

The solution set is {1.277}.

Caution Be careful when evaluating a quotient like $\frac{\ln 12}{\ln 7}$ in Example 1. Do not confuse this quotient with $\ln \frac{12}{7}$, which can be written as ln 12 – ln 7.

We cannot change the quotient of two logarithms to a difference of logarithms.

$$\frac{\ln 12}{\ln 7} \neq \ln \frac{12}{7}$$

Example 2 SOLVING AN EXPONENTIAL EQUATION

Solve $3^{2x-1} = .4^{x+2}$. Give the solution to the nearest thousandth.

Solution

$$3^{2x-1} = .4^{x+2}$$

$$\ln 3^{2x-1} = \ln .4^{x+2}$$ Take natural logarithms on both sides.

$$(2x-1)\ln 3 = (x+2)\ln .4$$ Property power

$$2x \ln 3 - \ln 3 = x \ln .4 + 2 \ln .4$$ Distributive property

Example 2 SOLVING AN EXPONENTIAL EQUATION

Solve $3^{2x-1} = .4^{x+2}$. Give the solution to the nearest thousandth.

Solution

$$2x \ln 3 - x \ln .4 = 2 \ln .4 + \ln 3$$ Write the terms with x on one side

$$x(2 \ln 3 - \ln .4) = 2 \ln .4 + \ln 3$$ Factor out x.

$$x = \frac{2 \ln .4 + 3}{2 \ln 3 - .4}$$ Divide by $2 \ln 3 - \ln .4$.

$$x = \frac{\ln .4^2 + \ln 3}{\ln 3^2 - \ln .4}$$ Power property

Example 2 SOLVING AN EXPONENTIAL EQUATION

Solve $3^{2x-1} = .4^{x+2}$. Give the solution to the nearest thousandth.

Solution

$$x = \frac{\ln .16 + \ln 3}{\ln 9 - \ln .4}$$ Apply the exponents.

This is exact.

$$x = \frac{\ln .48}{\ln \frac{9}{.4}}$$ Product property; Quotient property

$$x \approx -.236$$ This is approximate.

The solution set is $\{-.236\}$.

Example 3 SOLVING BASE *e* EXPONENTIAL EQUATIONS

Solve the equation. Give solutions to the nearest thousandth.

a. $e^{x^2} = 200$

Solution

$$e^{x^2} = 200$$

$$\ln e^{x^2} = \ln 200$$ Take natural logarithms on both sides.

$$x^2 = \ln 200$$ $\ln e^{x^2} = x^2$

Example 3 SOLVING BASE *e* EXPONENTIAL EQUATIONS

Solve the equation. Give solutions to the nearest thousandth.

a. $e^{x^2} = 200$

Remember both roots.

Solution

$$x = \pm\sqrt{\ln 200}$$ Square root property

$$x \approx \pm 2.302$$ Use a calculator.

The solution set is $\{\pm 2.302\}$.

Example 3 SOLVING BASE *e* EXPONENTIAL EQUATIONS

Solve the equation. Give solutions to the nearest thousandth.

b. $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

$$e^{2x+1} \cdot e^{-4x} = 3e$$

$$e^{-2x+1} = 3e$$ $a^m \cdot a^n = a^{m+n}$

$$e^{-2x} = 3$$ Divide by e; $\frac{a^m}{a^n} = a^{m-n}$.

$$\ln e^{-2x} = \ln 3$$ Take natural logarithms on both sides.

$$-2x \ln e = \ln 3$$ Power property

Example 3 SOLVING BASE *e* EXPONENTIAL EQUATIONS

Solve the equation. Give solutions to the nearest thousandth.

b. $e^{2x+1} \bullet e^{-4x} = 3e$

Solution

$$-2x = \ln 3 \quad \text{In } e = 1$$

$$x = -\frac{1}{2}\ln 3 \quad \text{Multiply by } -\tfrac{1}{2}$$

$$x \approx -.549$$

The solution set is {–.549}.

Example 4 SOLVING A LOGARITHMIC EQUATION

Solve log(x + 6) – log(x + 2) = log x.

Solution

$$\log(x+6) - \log(x+2) = \log x$$

$$\log\frac{x+6}{x+2} = \log x \quad \text{Quotient property}$$

$$\frac{x+6}{x+2} = x \quad \text{Property of logarithms}$$

$$x + 6 = x(x+2)$$

Example 4 SOLVING A LOGARITHMIC EQUATION

Solve log(x + 6) – log(x + 2) = log x.

Solution

$$x + 6 = x^2 + 2x \quad \text{Distributive property}$$

$$x^2 + x - 6 = 0 \quad \text{Standard form}$$

$$(x+3)(x-2) = 0 \quad \text{Factor.}$$

$$x = -3 \quad \text{or} \quad x = 2 \quad \text{Zero-factor property}$$

The proposed negative solution (x = –3) is not in the domain of the log x in the original equation, so the only valid solution is the positive number 2, giving the solution set {2}.

Caution Recall that the domain of $y = \log_a x$ is $(0, \infty)$. For this reason, ***it is always necessary to check that proposed solutions of a logarithmic equation result in logarithms of positive numbers in the original equation.***

Example 5 SOLVING A LOGARITHMIC EQUATION

Solve log(3x + 2) + log(x – 1) = 1. Give the exact value(s) of the solution(s).

Solution

$$\log(3x+2) + \log(x-1) = 1$$

$$\log(3x+2) + \log(x-1) = \log 10 \quad \text{Substitute.}$$

$$\log[(3x+2)(x-1)] = \log 10 \quad \text{Product property}$$

$$(3x+2)(x-1) = 10 \quad \text{Property of logarithms}$$

Example 5 SOLVING A LOGARITMIC EQUATION

Solve (3x + 2) + log(x – 1) = 1. Give the exact value(s) of the solution(s).

Solution

$$3x^2 - x - 2 = 10 \quad \text{Multiply.}$$

$$3x^2 - x - 12 = 0 \quad \text{Subtract 10.}$$

$$x = \frac{1 \pm \sqrt{1+144}}{6} \quad \text{Quadratic formula}$$

Example 5 SOLVING A LOGARITMIC EQUATION

Solve (3x + 2) + log(x – 1) = 1. Give the exact value(s) of the solution(s).

Solution

The number $\frac{1-\sqrt{145}}{6}$ is negative, so x – 1 is negative. Therefore, log(x – 1) is not defined and this proposed solution must be discarded.

Since $\frac{1+\sqrt{145}}{6}$ > 1, both 3x + 2 and x – 1 are positive and the solution set is $\left\{\frac{1+\sqrt{145}}{6}\right\}$.

Note The definition of logarithm could have been used in Example 5 by first writing

$$\log(3x+2)+\log(x-1)=1$$

$$\log_{10}[(3x+2)(x-1)]=1 \quad \text{Product property}$$

$$(3x+2)(x-1)=10^1, \quad \text{Definition of logarithm}$$

and then continuing as shown above.

Example 6 SOLVING A BASE *e* LOGARITHMIC EQUATION

Solve ln $e^{\ln x}$ –ln(x – 3) = ln 2. Give the exact value(s) of the solution(s).

Solution

$$\ln e^{\ln x}-\ln(x-3)=\ln 2$$

$$\ln x-\ln(x-3)=\ln 2 \quad e^{\ln x}=x$$

$$\ln \frac{x}{x-3}=\ln 2 \quad \text{Quotient property}$$

$$\frac{x}{x-3}=2 \quad \text{Property of logarithms}$$

Example 6 SOLVING A BASE *e* LOGARITHMIC EQUATION

Solve ln $e^{\ln x}$ –ln(x – 3) = ln 2. Give the exact value(s) of the solution(s).

Solution

$$x=2x-6 \quad \text{Multiply by } x-3.$$

$$6=x \quad \text{Solve for } x.$$

Verify that the solution set is {6}.

Solving Exponential Or Logarithmic Equations

To solve an exponential or logarithmic equation, change the given equation into one of the following forms, where *a* and *b* are real numbers, *a* > 0 and *a* ≠ 1, and follow the guidelines.

1. $a^{f(x)} = b$
   Solve by taking logarithms on both sides.
2. $\log_a f(x) = b$
   Solve by changing to exponential form $a^b = f(x)$.

Solving Exponential Or Logarithmic Equations

3. $\log_a f(x) = \log_a g(x)$
   The given equation is equivalent to the equation $f(x) = g(x)$. Solve algebraically.
4. In a more complicated equation, such as the one in Example 3(b), it may be necessary to first solve for $a^{f(x)}$ or $\log_a f(x)$ and then solve the resulting equation using one of the methods given above.
5. Check that the proposed solution is in the domain.

Example 7 **APPLYING AN EXPONENTIAL EQUATION TO THE STRENGTH OF A HABIT**

The strength of a habit is a function of the number of times the habit is repeated. If N is the number of repetitions and H is the strength of the habit, then, according to psychologist C. L. Hull,

$$H = 1000(1 - e^{-kN}),$$

where k is a constant. Solve this equation for k.

Example 7 **APPLYING AN EXPONENTIAL EQUATION TO THE STRENGTH OF A HABIT**

Solution

First solve the equation for e^{-kN}.

$$H = 1000(1 - e^{-kN})$$

$$\frac{H}{1000} = 1 - e^{-kN}$$ Divide by 1000.

$$\frac{H}{1000} - 1 = -e^{-kN}$$ Subtract 1.

$$e^{-kN} = 1 - \frac{H}{1000}$$ Multiply by –1 and rewrite

Example 7 **APPLYING AN EXPONENTIAL EQUATION TO THE STRENGTH OF A HABIT**

Solution

Now we solve for k.

$$\ln e^{-kN} = \ln\left(1 - \frac{H}{1000}\right)$$ Take natural logarithms on both sides.

$$-kN = \ln\left(1 - \frac{H}{1000}\right)$$ $\ln e^x = x$

$$k = -\frac{1}{N}\ln\left(1 - \frac{H}{1000}\right)$$ Multiply by $-\frac{1}{N}$.

Example 7 **APPLYING AN EXPONENTIAL EQUATION TO THE STRENGTH OF A HABIT**

Solution

$$k = -\frac{1}{N}\ln\left(1 - \frac{H}{1000}\right)$$ Multiply by $-\frac{1}{N}$.

With the final equation, if one pair of values for H and N is known, k can be found, and the equation can then be used to find either H or N for given values of the other variable.

Example 8 **MODELING COAL CONSUMPTION IN THE U.S.**

The table gives U.S. coal consumption (in quadrillions of British thermal units, or *quads*) for several years. The data can be modeled with the function defined by

$$f(t) = 26.97 \ln t - 102.46, \quad t \geq 80,$$

Year	Coal Consumption (in quads)
1980	15.42
1985	17.48
1990	19.17
1995	20.09
2000	22.58
2005	22.39

where t is the number of years after 1900, and $f(t)$ is in quads.

Example 8 **MODELING COAL CONSUMPTION IN THE U.S.**

a. Approximately what amount of coal was consumed in the United States in 2003? How does this figure compare to the actual figure of 22.32 quads?

Solution

The year 2003 is represented by

t = 2003 – 1900 = 103.

$$f(103) = 26.97 \ln 103 - 102.46$$

$$\approx 22.54$$ Use a calculator.

Based on this model, 22.54 quads were used in 2003. This figure is very close to the actual amount of 22.32 quads.

Example 8 **MODELING COAL CONSUMPTION IN THE U.S.**

b. Let $f(t) = 25$, and solve for t.

Solution

$$25 = 26.97 \ln t - 102.46$$

$$127.46 = 26.97 \ln t \quad \text{Add 102.46}$$

$$\ln t = \frac{127.46}{26.97} \quad \text{Divide by 26.97; rewrite.}$$

$$t = e^{127.46/26.97} \quad \text{Write in exponential form.}$$

$$t \approx 113 \quad \text{Use a calculator.}$$

Add 113 to 1900 to get 2013. Annual consumption will reach 25 quads in approximately 2013.