10[TH] EDITION

COLLEGE ALGEBRA

LIAL
HORNSBY
SCHNEIDER

4.6 Applications and Models of Exponential Growth and Decay

- The Exponential Growth or Decay Function
- Growth Function Models
- Decay Function Models

Exponential Growth or Decay Function

In many situations that occur in ecology, biology, economics, and the social sciences, a quantity changes at a rate proportional to the amount present. In such cases the amount present at time t is a special function of t called an **exponential growth or decay function.**

Exponential Growth or Decay Function

Let y_0 be the amount or number present at time $t = 0$. Then, under certain conditions, the amount present at any time t is modeled by

$$y = y_0 e^{kt},$$

where k is a constant.

Exponential Growth or Decay Function

When $k > 0$, the function describes growth; in **Section 4.2,** we saw examples of exponential growth: compound interest and atmospheric carbon dioxide.

When $k < 0$, the function describes decay; one example of exponential decay is radioactivity.

Example 2 FINDING DOUBLING TIME FOR MONEY

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$$A = Pe^{rt}$$ Continuous compounding formula

$$2P = Pe^{.03t}$$ Let $A = 2P$ and $r = .03$

$$2 = e^{.03t}$$ Divide by P

$$\ln 2 = \ln e^{.03t}$$ Take logarithms on both sides.

Example 2 FINDING DOUBLING TIME FOR MONEY

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$\ln 2 = .03t$ — In $e^x = x$

$\frac{\ln 2}{.03} = t$ — Divide by .03

$23.10 \approx t$ — Use a calculator.

It will take about 23 yr for the amount to double.

Example 3 DETERMINING AN EXPONENTIAL FUNCTION TO MODEL POPULATION GROWTH

According to the U.S. Census Bureau, the world population reached 6 billion people on July 18, 1999, and was growing exponentially. By the end of 2000, the population had grown to 6.079 billion. The projected world population (in billions of people) t years after 2000, is given by the function defined by

$$f(t) = 6.079e^{.0126t}.$$

Example 3 DETERMINING AN EXPONENTIAL FUNCTION TO MODEL POPULATION GROWTH

a. Based on this model, what will the world population be in 2010?

Solution

Since $t = 0$ represents the year 2000, in 2010, t would be 2010 – 2000 = 10 yr. We must find $f(t)$ when t is 10.

$$f(t) = 6.079e^{.0126t}$$

$$f(10) = 6.079e^{(.0126)10}$$ Let $t = 10$.

$$\approx 6.895$$

According to the model, the population will be 6.895 billion at the end of 2010.

Example 3 DETERMINING AN EXPONENTIAL FUNCTION TO MODEL POPULATION GROWTH

b. In what year will the world population reach 7 billion?

Solution

$$f(t) = 6.079e^{.0126t}$$

$$7 = 6.079e^{.0126t}$$ Let $t = 7$.

$$\frac{7}{6.079} = e^{.0126t}$$ Divide by 6.079

$$\ln \frac{7}{6.079} = \ln e^{.0126t}$$ Take logarithms on both sides

Example 3 DETERMINING AN EXPONENTIAL FUNCTION TO MODEL POPULATION GROWTH

b. In what year will the world population reach 7 billion?

Solution

$$\ln \frac{7}{6.079} = .0126t$$ In $e^x = x$

$$t = \frac{\ln \frac{7}{6.079}}{.0126}$$ Divide by .0126; rewrite.

$$t \approx 11.2$$ Use a calculator.

World population will reach 7 billion 11.2 yr after 2000, during the year 2011.

Example 4 DETERMINING AN EXPONENTIAL FUNCTION TO MODEL RADIOACTIVE DECAY

If 600 g of a radioactive substance are present initially and 3 yr later only 300 g remain, how much of the substance will be present after 6 yr?

Solution

To express the situation as an exponential equation

$$y = y_0 e^{kt},$$

we use the given values to first find y_0 and then find k.

Example 4 **DETERMINING AN EXPONENTIAL FUNCTION TO MODEL RADIOACTIVE DECAY**

If 600 g of a radioactive substance are present initially and 3 yr later only 300 g remain, how much of the substance will be present after 6 yr?

Solution

$$600 = y_0 e^{k(0)}$$ Let $y = 600$ and $t = 0$.

$$600 = y_0$$ $e^0 = 1$

Thus, $y_0 = 600$ and the exponential equation $y = y_0 e^{kt}$ becomes

$$y = 600e^{kt}.$$

Example 4 **DETERMINING AN EXPONENTIAL FUNCTION TO MODEL RADIOACTIVE DECAY**

Solution

Now solve this exponential equation for *k.*

$$y = 600e^{kt}$$

$$300 = 600e^{3k}$$ Let $y = 300$ and $t = 3$.

$$.5 = e^{3k}$$ Divide by 600.

$$\ln .5 = \ln e^{3k}$$ Take logarithms on both sides.

Example 4 **DETERMINING AN EXPONENTIAL FUNCTION TO MODEL RADIOACTIVE DECAY**

Solution

$$\ln .5 = 3k$$ $\ln e^x = x$

$$\frac{\ln .5}{3} = k$$ Divide by 3.

$$k \approx -.231$$ Use a calculator.

Example 4 **DETERMINING AN EXPONENTIAL FUNCTION TO MODEL RADIOACTIVE DECAY**

Solution

Thus, the exponential decay equation is $y = 600e^{-.231t}$. To find the amount present after 6 yr, let $t = 6$.

$$y = 600e^{-.231(6)} \approx 600e^{-1.386} \approx 150$$

After 6 yr, about 150 g of the substance will remain.

Half Life

Analogous to the idea of doubling time is **half-life,** the amount of time that it takes for a quantity that decays exponentially to become half its initial amount.

Example 5 **SOLVING A CARBON DATING PROBLEM**

Carbon 14, also known as radiocarbon, is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radiocarbon disintegrates. Scientists can determine the age of the remains by comparing the amount of radiocarbon with the amount present in living plants and animals. This technique is called **carbon dating.** The amount of radiocarbon present after *t* years is given by

$$y = y_0 e^{-.0001216t}$$

where y_0 is the amount present in living plants and animals.

Example 5 **SOLVING A CARBON DATING PROBLEM**

a. Find the half-life.

Solution

If y_0 is the amount of radiocarbon present in a living thing, then ½ y_0 is half this initial amount. Thus, we substitute and solve the given equation for *t.*

$$y = y_0 e^{-.0001216t}$$

$$\frac{1}{2} y_0 = y_0 e^{-.0001216t} \qquad \text{Let } y = \tfrac{1}{2} y_0.$$

Example 5 **SOLVING A CARBON DATING PROBLEM**

a. Find the half-life.

Solution

$$\frac{1}{2} y_0 = y_0 e^{-.0001216t} \qquad \text{Let } y = \tfrac{1}{2} y_0.$$

$$\frac{1}{2} = e^{-.0001216t} \qquad \text{Divide by } y_0.$$

$$\ln \frac{1}{2} = \ln e^{-.0001216t} \qquad \text{Take logarithms on both sides.}$$

Example 5 **SOLVING A CARBON DATING PROBLEM**

a. Find the half-life.

Solution

$$\ln \frac{1}{2} = -.0001216t \qquad \ln e^x = x$$

$$\frac{\ln \frac{1}{2}}{-.0001216} = t \qquad \text{Divide by } -.0001216$$

$$5700 \approx t \qquad \text{Use a calculator.}$$

The half-life is about 5700 yr.

Example 5 **SOLVING A CARBON DATING PROBLEM**

b. Charcoal from an ancient fire pit on Java contained ¼ the carbon 14 of a living sample of the same size. Estimate the age of the charcoal.

Solution

Solve again for *t,* this time letting the amount $y = \frac{1}{4} y_0$.

$$y = y_0 e^{-.0001216t} \qquad \text{Given equation.}$$

$$\frac{1}{4} y_0 = y_0 e^{-.0001216t} \qquad \text{Let } y = \tfrac{1}{4} y_0.$$

Example 5 **SOLVING A CARBON DATING PROBLEM**

Solution

$$\frac{1}{4} = e^{-.0001216t} \qquad \text{Divide by } y_0.$$

$$\ln \frac{1}{4} = \ln e^{-.0001216t} \qquad \text{Take logarithms on both sides}$$

$$\frac{\ln \frac{1}{4}}{-.0001216} = t \qquad \ln e^x = x\text{; divide by } -.0001216$$

$$t \approx 11{,}400 \qquad \text{Use a calculator.}$$

The charcoal is about 11,400 yr old.

Example 6 **MODELING NEWTON'S LAW OF COOLING**

Newton's law of cooling says that the rate at which a body cools is proportional to the difference *C* in temperature between the body and the environment around it. The temperature $f(t)$ of the body at time *t* in appropriate units after being introduced into an environment having constant temperature T_0 is

$$f(t) = T_0 + Ce^{-kt},$$

where *C* and *k* are constants.

Example 6 MODELING NEWTON'S LAW OF COOLING

A pot of coffee with a temperature of 100°C is set down in a room with a temperature of 20°C. The coffee cools to 60°C after 1 hr.

a. Write an equation to model the data.

Solution

We must find values for C and k in the formula for cooling. From the given information, when $t = 0$, $T_0 = 20$, and the temperature of the coffee is $f(0) = 100$. Also, when $t = 1$, $f(1) = 60$. Substitute the first pair of values into the equation along with $T_0 = 20$.

$$f(t) = T_0 + Ce^{-kt},$$

Example 6 MODELING NEWTON'S LAW OF COOLING

a. Write an equation to model the data.

Solution

$$f(t) = T_0 + Ce^{-kt}$$ Given formula

$$100 = 20 + Ce^{-0k}$$ Let $t = 0$, $f(0) = 100$, and $T_0 = 20$.

$$100 = 20 + C$$ $e^0 = 1$

$$80 = C$$ Subtract 20.

Thus, $f(t) = 20 + 80e^{-kt}$.

Example 6 MODELING NEWTON'S LAW OF COOLING

Solution

Now use the remaining pair of values in this equation to find k.

$$f(t) = T_0 + 80e^{-kt}$$ Given formula

$$60 = 20 + 80e^{-1k}$$ Let $t = 1$, $f(1) = 60$.

$$40 = 80e^{-k}$$ Subtract 20.

$$\frac{1}{2} = e^{-k}$$ Divide by 80.

$$\ln\frac{1}{2} = \ln e^{-k}$$ Take logarithms on both sides.

Example 6 MODELING NEWTON'S LAW OF COOLING

Solution

Now use the remaining pair of values in this equation to find k.

$$\ln\frac{1}{2} = -k$$ $\ln e^x = x$

$$k = -\ln\frac{1}{2} \approx .693$$

Thus, $f(t) = 20 + 80e^{-.693t}$.

Example 6 MODELING NEWTON'S LAW OF COOLING

b. Find the temperature after half an hour.

Solution

To find the temperature after ½ hr, let $t = ½$ in the model from part (a).

$$f(t) = 20 + 80e^{-.693t}$$ Model from part (a)

$$f\left(\frac{1}{2}\right) = 20 + 80e^{(-.693)(1/2)} \approx 76.6°C$$ Let $t = ½$.

Example 6 MODELING NEWTON'S LAW OF COOLING

c. How long will it take for the coffee to cool to 50°C?

Solution

To find how long it will take for the coffee to cool to 50°C, let $f(t) = 50$.

$$50 = 20 + 80e^{-.693t}$$ Let $f(t) = 50$.

$$30 = 80e^{-.693t}$$ Subtract 20.

$$\frac{3}{8} = e^{-.693t}$$ Divide by 80.

Example 6 **MODELING NEWTON'S LAW OF COOLING**

c. How long will it take for the coffee to cool to 50°C?

Solution

To find how long it will take for the coffee to cool to 50°C, let $f(t) = 50$.

$$\ln\frac{3}{8} = \ln e^{-.693t}$$ Take logarithms on both sides.

$$\ln\frac{3}{8} = -.693t$$ $\ln e^x = x$

$$t = \frac{\ln\frac{3}{8}}{-.693} \approx 1.415 \text{ hr, or about 1 hr 25 min}$$