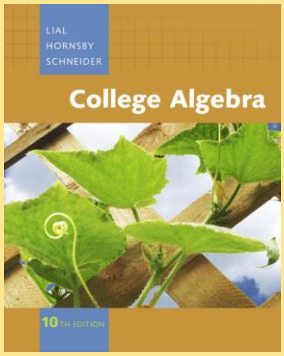
4

Inverse, Exponential, and Logarithmic Functions



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4 Inverse, Exponential, and Logarithmic Functions

4.1 Inverse Functions

Inverse Operations ▪ One-to-One Functions ▪ Inverse Functions ▪ Equations of Inverses ▪ An Application of Inverse Functions to Cryptography

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4.1 **Example 1** Deciding Whether Functions are One-to-One (page 403)

Decide whether each function is one-to-one.

(a) $f(x) = -3x + 7$

We must show that $f(a) = f(b)$ leads to the result $a = b$.

$$f(a) = f(b) \Rightarrow -3a + 7 = -3b + 7 \Rightarrow -3a = -3b \Rightarrow a = b.$$

$f(x) = -3x + 7$ is one-to-one.

4.1 **Example 1** Deciding Whether Functions are One-to-One (cont.)

Decide whether each function is one-to-one.

(b) $f(x) = \sqrt{49 - x^2}$

If we choose $a = 7$ and $b = -7$, then $7 \neq -7$, but

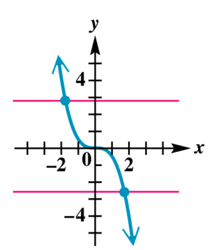
$f(7) = \sqrt{49 - 7^2} = 0$ and $f(-7) = \sqrt{49 - (-7)^2} = 0.$

So, even though $7 \neq -7$, $f(7) = f(-7) = 0$.

$f(x) = \sqrt{49 - x^2}$ is not one-to-one.

4.1 **Example 2(a)** Using the Horizontal Line Test (page 404)

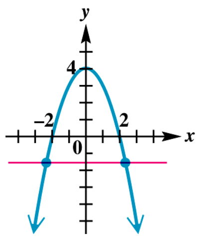
Determine whether the graph is the graph of a one-to-one function.



Since every horizontal line will intersect the graph at exactly one point, the function is one-to-one.

4.1 **Example 2(a)** Using the Horizontal Line Test (page 404)

Determine whether the graph is the graph of a one-to-one function.



Since the horizontal line will intersect the graph at more than one point, the function is not one-to-one.

4.1 **Example 3** Deciding Whether Two Functions are Inverses (page 405)

Let $f(x) = 2x + 5$ and $g(x) = \frac{1}{2}x - 5$. Is g the inverse function of f?

$f(x) = 2x + 5$ is a nonhorizontal linear function.

Thus, f is one-to-one, and it has an inverse.

Now find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = 2\left(\tfrac{1}{2}x - 5\right) + 5 = x - 10 + 5 = x - 5$$

$$(g \circ f)(x) = \tfrac{1}{2}(2x + 5) - 5 = x + \tfrac{5}{2} - 5 = x - \tfrac{5}{2}$$

Since $(f \circ g)(x) \neq (g \circ f)(x) \neq x$, g is not the inverse of f.

4.1 **Example 4(a)** Finding Inverses of One-to-One Functions (page 407)

Find the inverse of the function
F = {(–2, –8), (–1, –1), (0, 0), (1, 1), (2, 8)} .

F is one-to-one and has an inverse since each x-value corresponds to only one y-value and each y-value corresponds to only one x-value.

Interchange the x- and y-values in each ordered pair in order to find the inverse function.

$$F^{-1} = \{(-8,-2),(-1,-1),(0,0),(1,1),(8,2)\}$$

4.1 **Example 4(b)** Finding Inverses of One-to-One Functions (page 407)

Find the inverse of the function
G = {(–2, 5), (–1, 2), (0, 1), (1, 2), (2, 5)} .

Each x-value in G corresponds to just one y-value.

However, the y-value 5 corresponds to two x-values, –2 and 2.

Thus, G is not one-to-one and does not have an inverse.

4.1 **Example 4(c)** Finding Inverses of One-to-One Functions (page 407)

Find the inverse of the function h defined by the table.

Year	Number of Unhealthy Days
2000	25
2001	40
2002	33
2003	19
2004	7
2005	33

Source: Illinois Environmental Protection Agency.

Each x-value in h corresponds to just one y-value.

However, the y-value 33 corresponds to two x-values, 2002 and 2005.

Thus, h is not one-to-one and does not have an inverse.

4.1 **Example 5(a)** Finding Equations of Inverses (page 407)

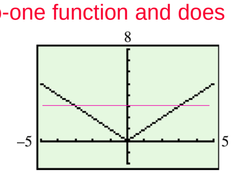
Is $f(x) = |x|$ a one-to-one function? If so, find the equation of its inverse.

If $x = 3, f(x) = |3| = 3.$

If $x = -3, f(x) = |-3| = 3.$

$f(x) = |x|$ is not a one-to-one function and does not have an inverse.

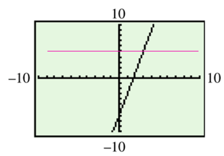
The horizontal line test confirms this.



4.1 **Example 5(b)** Finding Equations of Inverses (page 407)

Is $g(x) = 4x - 7$ a one-to-one function? If so, find the equation of its inverse.

The graph of g is a nonhorizontal line, so by the horizontal line test, g is a one-to-one function.



4.1 **Example 5(b)** Finding Equations of Inverses (cont.)

$y = 4x - 7$ — $y = f(x)$

$x = 4y - 7$ — *Step 1*: Interchange x and y.

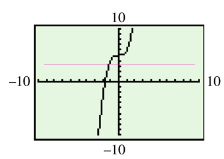
$y = \dfrac{x+7}{4}$ — *Step 2*: Solve for y.

$g^{-1}(x) = \dfrac{1}{4}x + \dfrac{7}{4}$ — *Step 3*: Replace y with $g^{-1}(x)$.

4.1 **Example 5(c)** Finding Equations of Inverses (page 407)

Is $h(x) = x^3 + 5$ a one-to-one function? If so, find the equation of its inverse.

A cubing function is one-to-one.



4.1 **Example 5(c)** Finding Equations of Inverses (cont.)

$y = x^3 + 5$ — $y = f(x)$

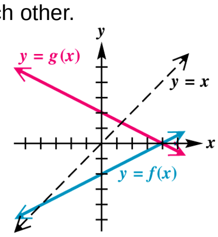
$x = y^3 + 5$ — *Step 1*: Interchange x and y.

$y = \sqrt[3]{x-5}$ — *Step 2*: Solve for y.

$h^{-1}(x) = \sqrt[3]{x-5}$ — *Step 3*: Replace y with $h^{-1}(x)$.

4.1 **Example 6** Graphing the Inverse (page 409)

Determine whether functions f and g graphed are inverses of each other.

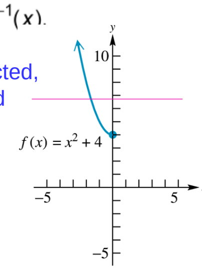


f and g graphed are not inverses because the graphs are not reflections across the line $y = x$.

4.1 **Example 7** Finding the Inverse of a Function with a Restricted Domain (page 409)

Let $f(x) = x^2 + 4, x \le 0$. Find $f^{-1}(x)$.

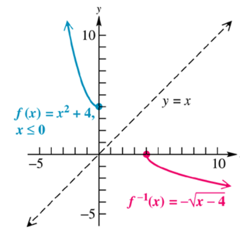
Because the domain is restricted, the function is one-to-one and has an inverse.



4.1 Example 7 Finding the Inverse of a Function with a Restricted Domain (cont.)

$y = x^2 + 4, x \le 0$	$y = f(x)$
$x = y^2 + 4, y \le 0$	*Step 1*: Interchange *x* and *y*.
$y = -\sqrt{x-4}$	*Step 2*: Solve for *y*. The domain of *f* is the range of $f^{-1}(x)$.
$f^{-1}(x) = -\sqrt{x-4}$	*Step 3*: Replace *y* with $f^{-1}(x)$.

4.1 Example 7 Finding the Inverse of a Function with a Restricted Domain (cont.)



f and f^{-1} are mirror images with respect to the line *y* = *x*.

4.1 Example 8 Using Functions to Encode and Decode a Message (page 410)

The function defined by *f*(*x*) = 3*x* – 1 was used to encode a message as

26 35 26 32 14 38 2 59 23

Find the inverse functions and decode the message. Use the values in the chart below.

A	1	H	8	O	15	V	22
B	2	I	9	P	16	W	23
C	3	J	10	Q	17	X	24
D	4	K	11	R	18	Y	25
E	5	L	12	S	19	Z	26
F	6	M	13	T	20		
G	7	N	14	U	21		

4.1 Example 8 Using Functions to Encode and Decode a Message (cont.)

The graph of *f*(*x*) = 3*x* – 1 is a nonhorizontal line, so by the horizontal line test, *f* is a one-to-one function and has an inverse.

$y = 3x - 1$	$y = f(x)$
$x = 3y - 1$	*Step 1*: Interchange *x* and *y*.
$y = \frac{x+1}{3} = \frac{1}{3}x + \frac{1}{3}$	*Step 2*: Solve for *y*.
$f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$	*Step 3*: Replace *y* with $f^{-1}(x)$.

4.1 Example 8 Using Functions to Encode and Decode a Message (cont.)

Use the inverse function $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$ to decode the message.

26 $f^{-1}(26) = \frac{1}{3}(26) + \frac{1}{3} = \frac{27}{3} = 9$; I	35 $f^{-1}(35) = \frac{1}{3}(35) + \frac{1}{3} = \frac{36}{3} = 12$; L	26 $f^{-1}(26) = \frac{1}{3}(26) + \frac{1}{3} = \frac{27}{3} = 9$; I
32 $f^{-1}(32) = \frac{1}{3}(32) + \frac{1}{3} = \frac{33}{3} = 11$; K	14 $f^{-1}(14) = \frac{1}{3}(14) + \frac{1}{3} = \frac{15}{3} = 5$; E	38 $f^{-1}(38) = \frac{1}{3}(38) + \frac{1}{3} = \frac{39}{3} = 13$; M
2 $f^{-1}(2) = \frac{1}{3}(2) + \frac{1}{3} = \frac{3}{3} = 1$; A	59 $f^{-1}(59) = \frac{1}{3}(59) + \frac{1}{3} = \frac{60}{3} = 20$; T	23 $f^{-1}(23) = \frac{1}{3}(23) + \frac{1}{3} = \frac{24}{3} = 8$; H

I LIKE MATH

4.2 Exponential Functions

Exponents and Properties ▪ Exponential Functions ▪ Exponential Equations ▪ Compound Interest ▪ The Number *e* and Continuous Compounding ▪ Exponential Models and Curve Fitting

4.2 **Example 1** Evaluating an Exponential Expression (page 417)

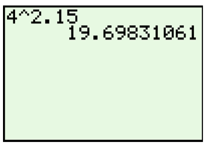
If $f(x) = 4^x$, find each of the following.

(a) $f(-2) = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

(b) $f(5) = 4^5 = 1024$

(c) $f\left(\frac{2}{3}\right) = 4^{2/3} = \left(4^2\right)^{1/3} = 16^{1/3} = \sqrt[3]{16} = \sqrt[3]{2 \cdot 2^3} = 2\sqrt[3]{2}$

(d) $f(2.15) = 4^{2.15} \approx 19.69831061$



4.2 **Example 2** Graphing an Exponential Function (page 419)

Graph $f(x) = \left(\frac{1}{2}\right)^x$.

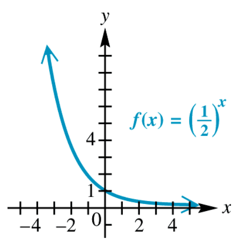
$f(0) = \left(\frac{1}{2}\right)^0 = 1$, so the y-intercept is 1. The x-axis is a horizontal asymptote.

f has domain $(-\infty, \infty)$ and range $(0, \infty)$ and is one-to-one.

4.2 **Example 2** Graphing an Exponential Function (cont.)

Create a table of values.

x	$f(x) = \left(\frac{1}{2}\right)^x$
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



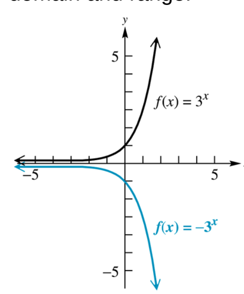
4.2 **Example 3(a)** Graphing Reflections and Translations (page 420)

Graph $f(x) = -3^x$. Give the domain and range.

The graph of $f(x) = -3^x$ is obtained by reflecting the graph of $f(x) = 3^x$ across the x-axis.

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0)$



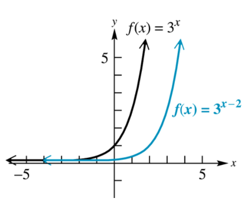
4.2 **Example 3(b)** Graphing Reflections and Translations (page 420)

Graph $f(x) = 3^{x-2}$. Give the domain and range.

The graph of $f(x) = 3^{x-2}$ is obtained by translating the graph of $f(x) = 3^x$ two units to the right.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



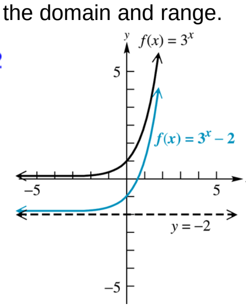
4.2 **Example 3(c)** Graphing Reflections and Translations (page 420)

Graph $f(x) = 3^x - 2$. Give the domain and range.

The graph of $f(x) = 3^x - 2$ is obtained by translating the graph of $f(x) = 3^x$ two units down.

Domain: $(-\infty, \infty)$

Range: $(-2, \infty)$



4.2 Example 4 Using a Property of Exponents to Solve an Equation (page 420)

Solve $5^x = \frac{1}{125}$.

$5^x = \frac{1}{125}$

$5^x = 125^{-1}$ Definition of negative exponent

$5^x = (5^3)^{-1}$ Write 125 as a power of 5.

$5^x = 5^{-3}$ $a^{mn} = (a^m)^n$

$x = -3$ Set exponents equal.

Solution set: {–3}

4.2 Example 5 Using a Property of Exponents to Solve an Equation (page 421)

Solve $3^{x+1} = 9^{x-3}$.

$3^{x+1} = 9^{x-3}$

$3^{x+1} = (3^2)^{x-3}$ Write 9 as a power of 3.

$3^{x+1} = 3^{2x-6}$ $(a^m)^n = a^{mn}$

$x+1 = 2x-6$ Set exponents equal.

$7 = x$ Solve for x.

Solution set: {7}

4.2 Example 6 Using a Property of Exponents to Solve an Equation (page 421)

Solve $b^{5/2} = 243$.

$b^{5/2} = 243$

$(b^{5/2})^{2/5} = 243^{2/5}$ Raise both sides to the 2/5 power.

$b = (\sqrt[5]{243})^2$ $(a^m)^n = a^{mn}$; $243^{2/5} = (\sqrt[5]{243})^2$

$b = \sqrt[5]{(3^5)^2}$ Write 243 as a power of 5.

$b = 3^2 = 9$ Simplify.

4.2 Example 6 Using a Property of Exponents to Solve an Equation (cont.)

It is necessary to check all proposed solutions in the original equation when both sides have been raised to a power.

Check $b = 9$.

$b^{5/2} = 243$

$9^{5/2} \stackrel{?}{=} 243$

$(\sqrt{9})^5 \stackrel{?}{=} 243$

$3^5 = 243 \Rightarrow 243 = 243$

Solution set: {9}

4.2 Example 7(a) Using the Compound Interest Formula (page 422)

Suppose $2500 is deposited in an account paying 6% per year compounded semiannually (twice per year). Find the amount in the account after 10 years with no withdrawals.

$A = P\left(1+\frac{r}{n}\right)^{tn}$ Compound interest formula

$A = (2500)\left(1+\frac{.06}{2}\right)^{10(2)}$ $P = 2500$, $r = .06$, $n = 2$, $t = 10$

$= (2500)(1.03)^{20}$

≈ 4515.28 Round to the nearest hundredth.

There is $4515.28 in the account after 10 years.

4.2 Example 7(b) Using the Compound Interest Formula (page 422)

How much interest is earned over the 10-year period?

The interest earned over the 10 years is $4515.28 – $2500 = $2015.28

4.2 Example 8(a) Finding Present Value (page 423)

Leah must pay a lump sum of $15,000 in 8 years. What amount deposited today at 4.8% compounded annually will give $15,000 in 8 years?

$A = P\left(1+\frac{r}{n}\right)^{tn}$	Compound interest formula
$15{,}000 = P\left(1+\frac{.048}{1}\right)^{8(1)}$	A = 15,000, r = .048, n = 1, t = 8
$\frac{15{,}000}{(1.048)^8} = P$	Simplify, then solve for P.
$P \approx 10{,}308.63$	Round to the nearest hundredth.

If Leah deposits $10,308.63 now, she will have $15,000 when she needs it.

4.2 Example 8(b) Finding Present Value (page 423)

If only $10,000 is available to deposit now, what annual interest rate is necessary for the money to increase to $15,000 in 8 years?

$A = P\left(1+\frac{r}{n}\right)^{tn}$	Compound interest formula
$15{,}000 = 10{,}000\left(1+\frac{r}{1}\right)^{8(1)}$	A = 15,000, P = 10,000, n = 1, t = 8
$\frac{3}{2} = (1+r)^8 \Rightarrow \left(\frac{3}{2}\right)^{1/8} - 1 = r$	Simplify, then solve for r.
$r \approx .05198$	Use a calculator.

An interest rate of about 5.20% will produce enough interest to increase the $10,000 to $15,000 by the end of 8 years.

4.2 Example 9 Solving a Continuous Compounding Problem (page 424)

Suppose $8000 is deposited in an account paying 5% interest compounded continuously for 6 years. Find the total amount on deposit at the end of 6 years.

$A = Pe^{rt}$	Continuous compounding formula
$A = 8000e^{.05(6)}$	P = 8000, r = .05, t = 6
$= 8000e^{.3}$	
$\approx 10{,}798.87$	Round to the nearest hundredth.

There will be about $10,798.87 in the account at the end of 6 years.

4.2 Example 10 Comparing Interest Earned as Compounding is More Frequent (page 425)

Suppose $2500 is invested at 6% in an account for 10 years. Find the amounts in the account at the end of 10 years if the interest is compounded quarterly, monthly, daily, and continuously.

Compounded	Formula
Quarterly	$A = 2500\left(1+\frac{.06}{4}\right)^{10(4)} \approx \$4535.05$
Monthly	$A = 2500\left(1+\frac{.06}{12}\right)^{10(12)} \approx \$4548.49$
Daily	$A = 2500\left(1+\frac{.06}{365}\right)^{10(365)} \approx \$4555.07$
Continuously	$A = 2500e^{.06(10)} \approx \$4555.30$

4.2 Example 11(a) Using Data to Model Exponential Growth (page 426)

If current trends of burning fossil fuels and deforestation continue, then future amounts of atmospheric carbon dioxide in parts per million (ppm) will increase as shown in the table.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC), 1990.

The data can be modeled by the function

$$y = 353e^{.0060857(t-1990)}$$

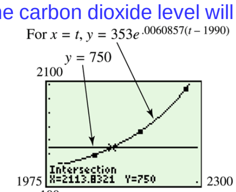
What will be the atmospheric carbon dioxide level in 2015?

$$y = 353e^{.0060857(2015-1990)} \approx 441 \text{ ppm}$$

4.2 Example 11(b) Using Data to Model Exponential Growth (page 426)

Use a graph of this model to estimate when the carbon dioxide level will be double the level that it was in 2000.

In 2000, the carbon dioxide level was 375 ppm, so we want to know when the carbon dioxide level will be 750 ppm.



The carbon dioxide level will be double the level in 2000, 750 ppm, by 2113.

4.3 Logarithmic Functions

Logarithms ▪ Logarithmic Equations ▪ Properties of Logarithms

4.3 Example 1 Solving Logarithmic Equations (page 433)

Solve:

(a) $\log_x \frac{16}{9} = -2$

$x^{-2} = \frac{16}{9}$

$x^2 = \frac{9}{16}$

$x = \sqrt{\frac{9}{16}} = \frac{3}{4}$

(b) $\log_{16} x = \frac{3}{4}$

$x = 16^{3/4} = \left(16^{1/4}\right)^3$

$x = 2^3 = 8$

(c) $\log_{36} \sqrt{6} = x$

$36^x = \sqrt{6}$

$\left(6^2\right)^x = 6^{1/2} \Rightarrow 6^{2x} = 6^{1/2}$

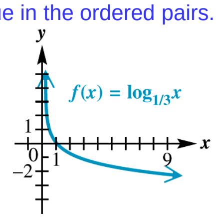
$2x = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

4.3 Example 2(a) Graphing Logarithmic Functions (page 436)

Graph $f(x) = y = \log_{1/3} x$.

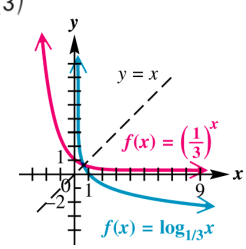
Use the exponential form $x = \left(\frac{1}{3}\right)^y$ to find ordered pairs. The ordered pairs for $y = \log_{1/3} x$ are found by interchanging the *x*- and *y*-value in the ordered pairs.

x	$y = \left(\frac{1}{3}\right)^x$	$y = \log_{1/3} x$
–2	9	undefined
–1	3	undefined
0	1	undefined
1	$\frac{1}{3}$	0
3	$\frac{1}{27}$	–1
9	$\frac{1}{19{,}683}$	–2



4.3 Example 2(a) Graphing Logarithmic Functions (cont.)

The graph of $f(x) = \log_{1/3} x$ is the reflection of the graph of $f(x) = \left(\frac{1}{3}\right)^x$ across the line $y = x$.

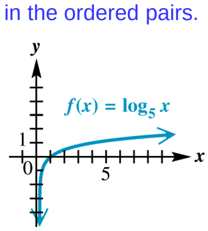


4.3 Example 2(b) Graphing Logarithmic Functions (page 436)

Graph $f(x) = \log_5 x$.

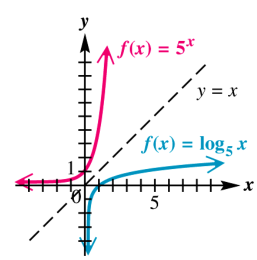
Use the exponential form $x = 5^y$ to find ordered pairs. The ordered pairs for $y = \log_5 x$ are found by interchanging the *x*- and *y*-value in the ordered pairs.

x	$y = 5^x$	$y = \log_5 x$
–2	$\frac{1}{25} = .04$	undefined
–1	$\frac{1}{5} = .2$	undefined
0	1	undefined
1	5	0
2	25	.43
3	125	.68



4.3 Example 2(b) Graphing Logarithmic Functions (cont.)

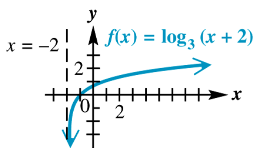
The graph of $f(x) = \log_5 x$ is the reflection of the graph of $f(x) = 5^x$ across the line $y = x$.



4.3 Example 3(a) Graphing Translated Logarithmic Functions (page 437)

Graph $f(x) = \log_3(x+2)$.

The graph of $f(x) = \log_3(x+2)$ is obtained by translating the graph of $f(x) = \log_3 x$ two units to the left. It has a vertical asymptote at $x = -2$.

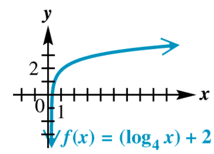


Domain: $(-2, \infty)$ Range: $(-\infty, \infty)$

4.3 Example 3(b) Graphing Translated Logarithmic Functions (page 437)

Graph $f(x) = (\log_4 x) + 2$.

The graph of $f(x) = (\log_4 x) + 2$ is obtained by translating the graph of $f(x) = \log_4 x$ two units up. It has a vertical asymptote at $x = 0$.



Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

4.3 Example 4 Using the Properties of Logarithms (page 438)

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_7(8 \cdot 6) = \log_7 8 + \log_7 6$

(b) $\log_6 \frac{12}{5} = \log_6 12 - \log_6 5$

(c) $\log_2 \sqrt[3]{9} = \log_2\left(9^{1/3}\right) = \frac{1}{3}\log_2 9$

(d) $\log_b \frac{rs^2t}{u^3v^5} = \log_b\left(rs^2t\right) - \log_b\left(u^3v^5\right)$

$= \log_b r + \log_b s^2 + \log_b t - \left(\log_b u^3 + \log_b v^5\right)$

$= \log_b r + 2\log_b s + \log_b t - 3\log_b u - 5\log_b v$

4.3 Example 4 Using the Properties of Logarithms (cont.)

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(e) $\log_a \sqrt[5]{r^3} = \log_a r^{3/5} = \frac{3}{5}\log_a r$

(f) $\log_a \sqrt[m]{\frac{r^3s^2}{t^4}} = \log_a\left(\frac{r^3s^2}{t^4}\right)^{1/m} = \frac{1}{m}\log_a\left(\frac{r^3s^2}{t^4}\right)$

$= \frac{1}{m}\log_a\left(r^3s^2\right) - \frac{1}{m}\log_a t^4$

$= \frac{1}{m}\log_a r^3 + \frac{1}{m}\log_a s^2 - \frac{1}{m}\log_a t^4$

$= \frac{3}{m}\log_a r + \frac{2}{m}\log_a s - \frac{4}{m}\log_a t$

4.3 Example 5 Using the Properties of Logarithms (page 439)

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_4 x - \log_4 y + \log_4 z = \log_4 \frac{xz}{y}$

(b) $4\log_b r - 5\log_b s = \log_b \frac{r^4}{s^5}$

(c) $\frac{1}{3}\log_a x + \frac{2}{3}\log_a y - \log_a xy$

$= \log_a \sqrt[3]{x} + \log_a \sqrt[3]{y^2} - \log_a xy$

$= \log_a \frac{\sqrt[3]{xy^2}}{xy}$

4.3 Example 6 Using the Properties of Logarithms (page 440)

Assume that $\log_{10} 7 = .8451$. Find each logarithm.

(a) $\log_{10} 49 = \log_{10} 7^2 = 2\log_{10} 7 = 2(.8451) = 1.6902$

(b) $\log_{10} 70 = \log_{10}(7 \cdot 10) = \log_{10} 7 + \log_{10} 10$

$= .8451 + 1 = 1.8451$

4.4 Evaluating Logarithms and the Change-of-Base Theorem

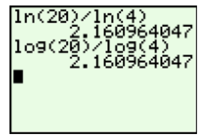
Common Logarithms ▪ Applications and Modeling with Common Logarithms ▪ Applications and Modeling with Natural Logarithms ▪ Logarithms with Other Bases

4.4 Example 6 Using the Change-of-Base Theorem (page 451)

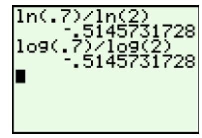
Use the change-of-base theorem to find an approximation to four decimal places for each logarithm.

(a) $\log_4 20 = \frac{\ln 20}{\ln 4} \approx 2.1610$ or $\log_4 20 = \frac{\log 20}{\log 4} \approx 2.1610$

(b) $\log_2 .7 = \frac{\ln .7}{\ln 2} \approx -.5146$ or $\log_2 .7 = \frac{\log .7}{\log 2} \approx -.5146$



(a)



(b)

4.4 Example 1(a) Finding pH (page 448)

Find the pH of a solution with $[H_3O^+] = 6.8 \times 10^{-8}$.

$$\begin{aligned} pH &= -\log[H_3O^+] \\ &= -\log(6.8 \times 10^{-8}) && \text{Substitute.} \\ &= -(\log 6.8 + \log 10^{-8}) && \text{Product property} \\ &= -(.8325 - 8) && \log 10^{-8} = -8 \\ &\approx 7.2 \end{aligned}$$

pH ≈ 7.2

4.4 Example 1(b) Finding pH (page 448)

Find the hydronium ion concentration of a solution with pH = 7.1.

$$\begin{aligned} pH &= -\log[H_3O^+] \\ 7.1 &= -\log[H_3O^+] && \text{Substitute.} \\ -7.1 &= \log[H_3O^+] && \text{Multiply by } -1. \\ [H_3O^+] &= 10^{-7.1} && \text{Write in exponential form.} \\ &\approx 5.0 \times 10^{-5} && \text{Use a calculator.} \end{aligned}$$

$[H_3O^+] \approx 5.0 \times 10^{-5}$

4.4 Example 2 Using pH in an Application (page 448)

Wetlands are classified as shown in the table.

Classification	pH
Bog	≤3.0
Poor fen	4.0–6.0
Rich fen	6.0–7.5

(Source: R. Mohlenbrock, "Summerby Swamp, Michigan," Natural History, March 1994.)

The hydronium ion concentration of a water sample from a wetland is 4.5 x 10^{-3}. Classify this wetland.

$$\begin{aligned} pH = -\log[H_3O^+] &= -\log(4.5 \times 10^{-3}) \\ &= -(\log 4.5 + \log 10^{-3}) \\ &= -(.6532 - 3) \approx 2.3 \end{aligned}$$

The wetland is a bog.

4.4 Example 3 Measuring the Loudness of Sound (page 449)

Find the decibel rating of a sound with intensity $10{,}000{,}000 I_0$.

$$\begin{aligned} d &= 10 \log \frac{I}{I_0} \\ d &= 10 \log \frac{10{,}000{,}000 I_0}{I_0} && \text{Let } I = 10{,}000{,}000 I_0. \\ &= 10 \log 10{,}000{,}000 \\ &= 10(7) = 70 && \log 10{,}000{,}000 = \log 10^7 = 7. \end{aligned}$$

The sound has a decibel rating of 70.

4.4 **Example 4(a)** Measuring the Age of Rocks (page 450)

By measuring the amounts of potassium 40, K, and argon 40, A, in a rock, the age t of the rock in years can be found with the formula

$$t = (1.26 \times 10^9)^{\frac{\ln\left[1+8.33\left(\frac{A}{K}\right)\right]}{\ln 2}}$$

How old is a rock sample in which $A = K$?

Since $A = K$, $\frac{A}{K} = 1$.

$$t = (1.26 \times 10^9)^{\frac{\ln[1+8.33(1)]}{\ln 2}} \approx 4.06 \times 10^9$$

The rock sample is about 4.06 billion years old.

4.4 **Example 4(b)** Measuring the Age of Rocks (page 450)

How old is a rock sample in which $\frac{A}{K} = .325$?

$$t = (1.26 \times 10^9)^{\frac{\ln\left[1+8.33\left(\frac{A}{K}\right)\right]}{\ln 2}}$$

$$t = (1.26 \times 10^9)^{\frac{\ln[1+8.33(.325)]}{\ln 2}}$$

$$\approx 2.38 \times 10^9$$

The rock sample is about 2.38 billion years old.

4.4 **Example 5** Modeling Global Temperature Increase (page 450)

Solar radiation trapped by carbon dioxide in the atmosphere (called radiative forcing) is measured in watts per square meter. Radiative forcing R can be modeled by

$$R = k \ln \frac{C}{C_0}$$

where C_0 is the preindustrial amount of carbon dioxide, C is the current carbon dioxide level, and k is a constant.

(a) Suppose that $C = 2C_0$ and $k = 14$. Find the radiative forcing under these conditions.

$$R = 14 \ln \frac{2C_0}{C_0} = 14 \ln 2 \approx 9.7 \text{ watts per m}^2$$

4.4 **Example 5** Modeling Global Temperature Increase (cont.)

(b) The average global temperature increase T (in °F) is given by $T(R) = 1.03R$. Find the average global temperature increase to the nearest degree Fahrenheit under the same conditions.

$$T(9.7) = 1.03(9.7) \approx 10°\text{F}$$

The average global temperature increase will be about 10°F.

4.4 **Example 7** Modeling Diversity of Species (page 452)

One measure of the diversity of the species in an ecological community is modeled by

$$H = -[P_1 \log_2 P_1 + P_2 \log_2 P_2 + \cdots + P_n \log_2 P_n]$$

where $P_1, P_2, \ldots, P_n$ are the proportions of a sample that belong to each of n species found in the sample.

(*Source:* Ludwig, J., and J. Reynolds, *Statistical Ecology: A Primer on Methods and Computing,* New York, Wiley, 1988, p. 92.)

4.4 **Example 7** Modeling Diversity of Species (cont.)

Find the measure of diversity in a community with two species where there are 60 of one species and 140 of the other.

There are 60 + 140 = 200 members in the community, so $P_1 = \frac{60}{200} = .3$ and $P_2 = \frac{140}{200} = .7$.

$$H = -[.3 \log_2 .3 + .7 \log_2 .7]$$

$$= -\left[.3 \frac{\ln .3}{\ln 2} + .7 \frac{\ln .7}{\ln 2}\right] \quad \text{Change-of-base theorem}$$

$$= -\left[\frac{.3 \ln .3 + .7 \ln .7}{\ln 2}\right] \approx .881$$

4.5 Exponential and Logarithmic Equations

Exponential Equations ▪ Logarithmic Equations ▪ Applications and Modeling

4.5 Example 1 Solving an Exponential Equation (page 458)

Solve $8^x = 21$. Give the solution to the nearest thousandth.

$$8^x = 21$$

$$\ln 8^x = \ln 21 \quad \text{Property of logarithms}$$

$$x \ln 8 = \ln 21 \quad \text{Power property}$$

$$x = \frac{\ln 21}{\ln 8} \approx 1.464 \quad \text{Divide by } \ln 8.$$

Solution set: {1.464}

4.5 Example 2 Solving an Exponential Equation (page 459)

Solve $5^{2x-3} = 8^{x+1}$. Give the solution to the nearest thousandth.

$$5^{2x-3} = 8^{x+1}$$

$$\ln\left(5^{2x-3}\right) = \ln\left(8^{x+1}\right) \quad \text{Property of logarithms}$$

$$(2x-3)\ln 5 = (x+1)\ln 8 \quad \text{Power property}$$

$$2x\ln 5 - 3\ln 5 = x\ln 8 + \ln 8 \quad \text{Distributive property}$$

$$2x\ln 5 - x\ln 8 = 3\ln 5 + \ln 8 \quad \text{Write the terms with } x \text{ on one side.}$$

$$x(2\ln 5 - \ln 8) = 3\ln 5 + \ln 8 \quad \text{Factor.}$$

$$x = \frac{3\ln 5 + \ln 8}{2\ln 5 - \ln 8} \approx 6.062$$

Solution set: {6.062}

4.5 Example 3(a) Solving Base *e* Exponential Equations (page 460)

Solve $e^{|x|} = 50$. Give the solution to the nearest thousandth.

$$e^{|x|} = 50$$

$$\ln e^{|x|} = \ln 50 \quad \text{Property of logarithms}$$

$$|x| = \ln 50 \quad \ln e^{|x|} = |x|$$

$$x \approx \pm 3.912$$

Solution set: {±3.912}

4.5 Example 3(b) Solving Base *e* Exponential Equations (page 460)

Solve $e^{4x} \cdot e^{x-1} = 5e$. Give the solution to the nearest thousandth.

$$e^{4x} \cdot e^{x-1} = 5e$$

$$e^{5x-1} = 5e \quad a^m \cdot a^n = a^{mn}$$

$$e^{5x-2} = 5 \quad \text{Divide by } e. \ \frac{a^m}{a^n} = a^{m-n}$$

$$\ln e^{5x-2} = \ln 5 \quad \text{Take natural logarithms on both sides.}$$

$$5x - 2 = \ln 5 \quad \ln e^{5x-2} = 5x-2$$

$$x = \frac{\ln 5 + 2}{5} \quad \text{Solve for } x.$$

$$\approx .722$$

Solution set: {.722}

4.5 Example 4 Solving an Logarithmic Equation (page 460)

Solve log(2*x* + 1) + log *x* = log(*x* + 8). Give the exact value(s) of the solution(s).

$$\log(2x+1) + \log x = \log(x+8)$$

$$\log\left[x(2x+1)\right] = \log(x+8) \quad \text{Product property}$$

$$\log\left(2x^2 + x\right) = \log(x+8) \quad \text{Distributive property}$$

$$2x^2 + x = x + 8 \quad \text{Property of logarithms}$$

$$2x^2 = 8 \quad \text{Solve the quadratic equation.}$$

$$x^2 = 4 \Rightarrow x = \pm 2$$

The negative solution is not in the domain of log *x* in the original equation, so the only valid solution is *x* = 2.

Solution set: {2}

4.5 **Example 5** Solving a Logarithmic Equation (page 461)

Solve $\log_2(2x-5)+\log_2(x-3)=3$. Give the exact value(s) of the solution(s).

$$\log_2(2x-5)+\log_2(x-3)=3$$

$$\log_2[(2x-5)(x-3)]=3 \quad \text{Product property}$$

$$(2x-5)(x-3)=2^3 \quad \text{Property of logarithms}$$

$$2x^2-11x+15=8 \quad \text{Multiply.}$$

$$2x^2-11x+7=0 \quad \text{Subtract 8.}$$

Use the quadratic formula with $a = 2$, $b = -11$, and $c = 7$ to solve for x.

4.5 **Example 5** Solving a Logarithmic Equation (cont.)

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-11)\pm\sqrt{(-11)^2-4(2)(7)}}{2(2)}=\frac{11\pm\sqrt{65}}{4}$$

The negative solution makes $2x - 5$ in the original equation negative, so reject that solution.

Solution set: $\left\{\frac{11+\sqrt{65}}{4}\right\}$

4.5 **Example 6** Solving a Logarithmic Equation (page 461)

Solve $\ln e^{\ln x}-\ln(x-4)=\ln 5$. Give the exact value(s) of the solution(s).

$$\ln e^{\ln x}-\ln(x-4)=\ln 5$$

$$\ln x-\ln(x-4)=\ln 5 \quad e^{\ln x}=x$$

$$\ln\frac{x}{x-4}=\ln 5 \quad \text{Quotient property}$$

$$\frac{x}{x-4}=5 \quad \text{Property of logarithms}$$

$$x=5x-20 \quad \text{Multiply by } x-4.$$

$$x=5 \quad \text{Solve for } x.$$

Solution set: {5}

4.5 **Example 7** Applying an Exponential Equation to the Strength of a Habit (page 462)

The equation $T=T_0+Ce^{-kt}$ can be used to describe Newton's law of cooling. Solve this equation for k.

$$T=T_0+Ce^{-kt}$$

$$T-T_0=Ce^{-kt}$$

$$\frac{T-T_0}{C}=e^{-kt}$$

$$\ln\left(\frac{T-T_0}{C}\right)=\ln e^{-kt} \quad \text{Take the natural logarithm of both sides.}$$

$$\ln\left(\frac{T-T_0}{C}\right)=-kt \quad \ln e^{-kt}=-kt$$

$$-\frac{1}{t}\ln\left(\frac{T-T_0}{C}\right)=k$$

4.5 **Example 8** Modeling Coal Consumption in the U.S. (page 463)

The table gives U.S. coal consumption (in quadrillions of British thermal units, or quads) for several years. The data can be modeled by

$$f(t)=26.97\ln t-102.46,\ t\geq 80$$

where t is the number of years after 1900, and $f(t)$ is in quads.

Year	Coal Consumption (in quads)
1980	15.42
1985	17.48
1990	19.17
1995	20.09
2000	22.58
2005	22.39

Source: Statistical Abstract of the United States.

4.5 **Example 8** Modeling Coal Consumption in the U.S. (cont.)

(a) Approximately what amount of coal was consumed in the United States in 1998? How does this figure compare to the actual figure of 21.66 quads?

The year 1998 is represented by $t = 1998 - 1900 = 98$.

$$f(98)=26.97\ln 98-102.46\approx 21.20 \text{ quads}$$

This is very close to the actual figure 21.66 quads.

4.5 **Example 8** Modeling Coal Consumption in the U.S. (cont.)

(b) If this trend continues, approximately when will annual consumption reach 28 quads?

Let $f(t) = 28$, and solve for t.

$$28 = 26.97\ln t - 102.46 \qquad f(t) = 26.97\ln t - 102.46,\ t \geq 80$$
$$130.46 = 26.97\ln t$$
$$\frac{130.46}{26.97} = \ln t$$
$$e^{130.46/26.97} = t \qquad \text{Write in exponential form.}$$
$$126.12 \approx t$$

Add 1900 to 126 to get 2026.

Annual consumption will reach 28 quads in 2026.

4.6 Applications and Models of Exponential Growth and Decay

The Exponential Growth or Decay Function ▪ Growth Function Models ▪ Decay Function Models

4.6 **Example 1** Determining an Exponential Function to Model the Increase of Carbon Dioxide (page 469)

The table shows the growth of atmospheric carbon dioxide over time.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC), 1990.

(a) Find an exponential model using the data for 2000 and 2175. Let the year 2000 correspond to $t = 0$.

The equation will take the form $y = y_0e^{kt}$. We must find the values of y_0 and k.

4.6 **Example 1** Determining an Exponential Function to Model the Increase of Carbon Dioxide (cont.)

The year 2000 corresponds to $t = 0$, so the year 2175 corresponds $t = 175$.

Since y_0 is the initial amount, $y_0 = 375$ when $t = 0$. Thus, the equation is $y = 375e^{kt}$.

When $t = 175$, $y = 1090$. Substitute these values into the equation, and solve for k.

$$1090 = 375e^{175k} \Rightarrow \frac{1090}{375} = e^{175k} \Rightarrow \ln\left(\frac{1090}{375}\right) = 175k$$
$$\Rightarrow \frac{\ln\frac{1090}{375}}{175} = k \Rightarrow .00610 \approx k$$

The equation of the model is $y = 375e^{.00610t}$.

4.6 **Example 1** Determining an Exponential Function to Model the Increase of Carbon Dioxide (cont.)

(b) Use the model to estimate when future levels of carbon dioxide will triple from the 1951 level of 280 ppm.

Let $y = 3(280) = 840$. Then solve for t.

$$840 = 375e^{.00610t} \Rightarrow \frac{840}{375} = e^{.00610t} \Rightarrow \ln\left(\frac{840}{375}\right) = .00610t$$
$$\Rightarrow \frac{\ln\frac{840}{375}}{.00610} = t \Rightarrow 132 \approx t$$

Since $t = 0$ corresponds to the year 2000, $t = 132$ corresponds to the year 2132.

The carbon dioxide level will triple from the 1951 level in 2132.

4.6 **Example 2** Finding Doubling Time for Money (page 470)

How long will it take for the money in an account that is compounded continuously at 5.75% to double?

Use the continuously compounding formula $A = Pe^{rt}$ with $A = 2P$ and $r = .0575$ to solve for t.

$$2P = Pe^{.0575t}$$
$$2 = e^{.0575t}$$
$$\ln 2 = .0575t \qquad \text{Take the natural logarithm of both sides.}$$
$$\frac{\ln 2}{.0575} \approx 12.05 \approx t$$

It will take about 12 years for the money to double.

4.6 **Example 3** Determining an Exponential Function to Model Population Growth (page 471)

The projected world population (in billions of people) t years after 2000, is given by the function $f(t) = 6.079e^{.0126t}$.

(a) What will the world population be at the end of 2015?

The year 2015 corresponds to $t = 15$.

$$f(t) = 6.079e^{.0126(15)} \approx 7.344$$

The world population will be about 7.344 billion at the end of 2015.

4.6 **Example 3** Determining an Exponential Function to Model Population Growth (cont.)

(b) In what year will the world population reach 8 billion?

Let $f(t) = 8$ and solve for t.

$$8 = 6.079e^{.0126t}$$

$$\frac{8}{6.079} = e^{.0126t}$$

$$\ln\left(\frac{8}{6.079}\right) = .0126t$$

$$\frac{\ln\frac{8}{6.079}}{.0126} \approx 21.8 \approx t$$

Since $t = 0$ corresponds to the year 2000, add 21 to 2000.

The world population reach 8 billion during the year 2021.

4.6 **Example 4** Determining an Exponential Function to Model Radioactive Decay (page 471)

If 800 grams of a radioactive substance are present initially, and 2.5 years later only 400 grams remain, how much of the substance will be present after 4 years?

The equation will take the form $y = y_0e^{kt}$. Let $y = 800$ and $t = 0$, then solve for y_0.

$$800 = y_0e^{k(0)} \Rightarrow 800 = y_0$$

The equation becomes $y = 800e^{kt}$. Now, let $y = 400$ and $t = 2.5$, then solve for k.

$$400 = 800e^{2.5k} \Rightarrow \frac{1}{2} = e^{2.5k} \Rightarrow \ln\left(\frac{1}{2}\right) = 2.5k \Rightarrow \frac{\ln\frac{1}{2}}{2.5} \approx -.277 \approx k$$

The exponential equation is $y = 800e^{-.277t}$.

4.6 **Example 4** Determining an Exponential Function to Model Radioactive Decay (cont.)

Letting $t = 4$, we have $y = 800e^{-.277(4)} \approx 264.2$.

There will be about 264 grams of the substance left after 4 years.

4.6 **Example 5** Solving a Carbon Dating Problem (page 472)

Suppose that the skeleton of a woman who lived in the Classical Greek period was discovered in 2005. Carbon 14 testing at that time determined that the skeleton contained ¾ of the carbon 14 of a living woman of the same size. Estimate the year in which the Greek woman died.

The amount of radiocarbon present after t years is given by $y = y_0e^{-.0001216t}$.

4.6 **Example 5** Solving a Carbon Dating Problem (cont.)

Let $y = \frac{3}{4}y_0$ and solve for t.

$$\frac{3}{4}y_0 = y_0e^{-.0001216t}$$

$$\frac{3}{4} = e^{-.0001216t}$$

$$\ln\left(\frac{3}{4}\right) = -.0001216t$$

$$\frac{\ln\frac{3}{4}}{-.0001216} \approx 2365.8 \approx t$$

The woman died approximately 2366 years before 2005, in 361 B.C.

4.6 Example 6 Modeling Newton's Law of Cooling (page 474)

Newton's law of cooling says that the rate at which a body cools is proportional to the difference C in temperature between the body and the environment around it. The temperature of the body at time t in appropriate units after being introduced into an environment having constant temperature T_0 is

$$f(t) = T_0 + Ce^{-kt}$$

where C and k are constants.

Newton's law of cooling also applies to warming.

4.6 Example 6 Modeling Newton's Law of Cooling (cont.)

Mollie took a leg of lamb out of her refrigerator, which is set at 34°F, and placed it in her oven, which she had preheated to 350°F. After 1 hour, her meat thermometer registered 70°F.

(a) Write an equation to model the data.

From the data, when $t = 0$, $f(0) = 34$ and $T_0 = 350$.

When $t = 1$, $f(1) = 70$.

Solve for C:

$$f(0) = 350 + Ce^{-k(0)} \Rightarrow 34 = 350 + C \Rightarrow C = -316$$

4.6 Example 6 Modeling Newton's Law of Cooling (cont.)

Now solve for k:

$$f(1) = 350 - 316e^{-k(1)}$$
$$70 = 350 - 316e^{-k(1)}$$
$$-280 = -316e^{-k}$$
$$\frac{280}{316} = e^{-k}$$
$$-\ln\left(\frac{280}{316}\right) \approx .1210 \approx k$$

The equation is $f(t) = 350 - 316e^{-.121t}$.

4.6 Example 6 Modeling Newton's Law of Cooling (cont.)

(b) Find the temperature 90 minutes after the leg of lamb was placed in the oven.

Time t is measured in hours, so convert 90 minutes to 1.5 hours.

$$f(1.5) = 350 - 316e^{-.121(1.5)} \approx 86.5$$

After 90 minutes (1.5 hours), the temperature is about 86.5°F.

4.6 Example 6 Modeling Newton's Law of Cooling (cont.)

(c) Mollie wants to serve the leg of lamb rare, which requires an internal temperature of 145°F. What is the total amount of time it will take to cook the leg of lamb?

Let $f(t) = 145$, then solve for t.

$$145 = 350 - 316e^{-.121t}$$
$$-205 = -316e^{-.121t}$$
$$\frac{205}{316} = e^{-.121t}$$
$$\ln\left(\frac{205}{316}\right) = -.121t \Rightarrow \frac{\ln\frac{205}{316}}{-.121} \approx 3.576 \approx t$$

The meat temperature will be 145°F after about 3.576 hours or about 3 hours 35 minutes.