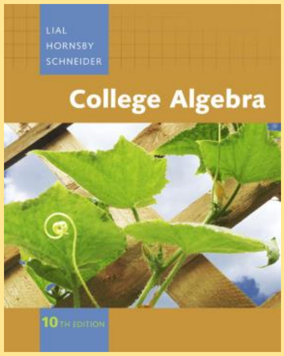
5

Systems and Matrices

Sections 5.1–5.5



---

5 Systems and Matrices

5.1 Systems of Linear Equations

Linear Systems ▪ Substitution Method ▪ Elimination Method ▪ Special Systems ▪ Applying Systems of Equations ▪ Solving Linear Systems with Three Unknowns (Variables) ▪ Using Systems of Equations to Model Data

---

5.1 Example 1 Solving a System by Substitution (page 495)

Solve the system.

$$4x - 3y = 14 \quad (1)$$
$$x - 2y = 1 \quad (2)$$

Solve equation (2) for x: $x = 1 + 2y$

Replace x in equation (1) with $1 + 2y$, then solve for y:

$$4(1 + 2y) - 3y = 14$$
$$4 + 8y - 3y = 14 \quad \text{Distributive property}$$
$$5y = 10$$
$$y = 2$$

Replace y in equation (2) with 2, then solve for y:

$$x - 2(2) = 1 \Rightarrow x - 4 = 1 \Rightarrow x = 5$$

5.1 Example 1 Solving a System by Substitution (cont.)

The solution of the system is (5, 2). **Check this solution in both equations (1) and (2).**

$4x - 3y = 14 \quad (1)$	$x - 2y = 1 \quad (2)$
$4(5) - 3(2) = 14 \quad ?$	$5 - 2(2) = 1 \quad ?$
$20 - 6 = 14$	$5 - 4 = 1$
$14 = 14$ True	$1 = 1$ True

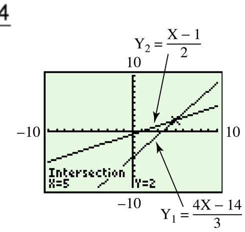
Solution set: {(5, 2)}

5.1 Example 1 Solving a System by Substitution (cont.)

To solve the system graphically, solve both equations for y:

$$4x - 3y = 14 \Rightarrow Y_1 = \frac{4X - 14}{3}$$
$$x - 2y = 1 \Rightarrow Y_2 = \frac{X - 1}{2}$$

Graph both Y_1 and Y_2 in the standard window to find that their point of intersection is (5, 2).



5.1 **Example 2** Solving a System by Elimination (page 496)

Solve the system.

$$\begin{aligned} 2x + 3y &= -1 \quad (1) \\ 3x - 2y &= 18 \quad (2) \end{aligned}$$

Multiply both sides of equation (1) by 2, and then multiply both sides of equation (2) by 3.

$$\begin{aligned} 4x + 6y &= -2 \quad (3) \\ 9x - 6y &= 54 \quad (4) \end{aligned}$$

Add equations (3) and (4), then solve for x.

$$\begin{aligned} 4x + 6y &= -2 \quad (3) \\ 9x - 6y &= 54 \quad (4) \\ \hline 13x \qquad &= 52 \Rightarrow x = 4 \end{aligned}$$

5.1 **Example 2** Solving a System by Elimination (cont.)

Substitute 4 for x equation (1), then solve for y.

$$\begin{aligned} 2x + 3y &= -1 \quad (1) \\ 2(4) + 3y &= -1 \\ 8 + 3y &= -1 \\ 3y &= -9 \\ y &= -3 \end{aligned}$$

The solution of the system is (4, –3). **Check this solution in both equations (1) and (2).**

5.1 **Example 2** Solving a System by Elimination (cont.)

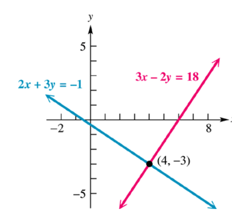
$$\begin{aligned} 2x + 3y &= -1 \quad (1) \\ 2(4) + 3(-3) &= -1 \quad ? \\ 8 - 9 &= -1 \\ -1 &= -1 \quad \text{True} \end{aligned}$$

$$\begin{aligned} 3x - 2y &= 18 \quad (2) \\ 3(4) - 2(-3) &= 18 \quad ? \\ 12 - (-6) &= 18 \\ 18 &= 18 \quad \text{True} \end{aligned}$$

Solution set: {(4, –3)}

5.1 **Example 2** Solving a System by Elimination (cont.)

The graph confirms that the solution set is {(4, –3)}.



5.1 **Example 3** Solving an Inconsistent System (page 496)

Solve the system.

$$\begin{aligned} 7x + 3y &= -5 \quad (1) \\ -14x - 6y &= -10 \quad (2) \end{aligned}$$

Multiply both sides of equation (1) by 2, then add the resulting equation to equation (2).

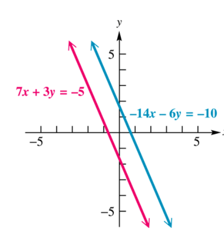
$$\begin{aligned} -14x + 6y &= -10 \quad (3) \\ -14x - 6y &= -10 \quad (2) \\ \hline 0 &= -20 \quad \text{False} \end{aligned}$$

The system is inconsistent.

Solution set: ø

5.1 **Example 3** Solving an Inconsistent System (cont.)

The graphs of the equations are parallel and never intersect.



5.1 **Example 4** Solving a System with Infinitely Many Solutions (page 497)

Solve the system.

$$\begin{aligned} -9x + 3y &= -24 \quad (1) \\ 3x - \ \ y &= 8 \quad (2) \end{aligned}$$

Multiply both sides of equation (2) by 3, then add the resulting equation to equation (1).

$$\begin{aligned} -9x + 3y &= -24 \quad (1) \\ 9x - 3y &= 24 \quad (2) \\ \hline 0 &= 0 \quad \text{True} \end{aligned}$$

The result indicates that the equations of the original system are equivalent. Any ordered pair that satisfies either equation will satisfy the system.

5.1 **Example 4** Solving a System with Infinitely Many Solutions (cont.)

From equation (2), we have

$$3x - y = 8 \Rightarrow y = 3x - 8$$

The solution set (with x arbitrary) is $\{(x, 3x - 8)\}$.

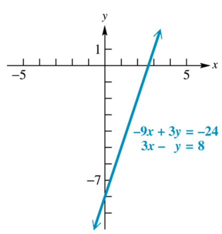
From equation (2), we have

$$3x - y = 8 \Rightarrow 3x = y + 8 \Rightarrow x = \frac{1}{3}y + \frac{8}{3}$$

The solution set (with y arbitrary) is $\left\{\left(\frac{1}{3}y + \frac{8}{3}, y\right)\right\}$.

5.1 **Example 4** Solving a System with Infinitely Many Solutions (cont.)

The graphs of the two equations coincide.



5.1 **Example 5** Using a Linear System to Solve an Application (page 498)

In the 2006 Winter Olympics in Torino, Italy, the two countries winning the most medals were Germany and the United States. The total number of medals won by these two countries was 54, with Germany winning 4 more medals than the United States. How many medals were won by each country? (*Source*: www.infoplease.com)

Let x = the number of medals won by the U.S.
Let y = the number of medals won by Germany.
Then, we have the system:

$$\begin{aligned} x + y &= 54 \quad (1) \\ y &= x + 4 \quad (2) \end{aligned}$$

5.1 **Example 5** Using a Linear System to Solve an Application (cont.)

Substitute $x + 4$ for y in equation (1), then solve for x.

$$\begin{aligned} x + (x + 4) &= 54 \quad x + y = 54 \ (1) \\ 2x + 4 &= 54 \\ 2x &= 50 \\ x &= 25 \end{aligned}$$

Substitute 25 for x in equation (2), then solve for y.

$$y = 25 + 4 = 29$$

The U.S. won 25 medals and Germany won 29 medals.

5.1 **Example 5** Using a Linear System to Solve an Application (cont.)

Verify that (25, 29) checks in the words of the original problem.

The total number of medals won was 25 + 29 = 54.

29 is 4 more than 25.

5.1 Example 6 Solving a System of Three Equations with Three Variables (page 500)

Solve the system

$$\begin{aligned} 3x + 4y - 2z &= 14 \quad (1)\\ 2x + y + 2z &= -9 \quad (2)\\ x - y + z &= -9 \quad (3) \end{aligned}$$

Eliminate y by adding equations (2) and (3) to obtain

$$3x + 3z = -18 \Rightarrow x + z = -6 \quad (4)$$

To eliminate y from another pair of equations, multiply equation (3) by 4 and add the result to equation (1).

$$\begin{aligned} 3x + 4y - 2z &= 14 \quad (1)\\ 4x - 4y + 4z &= -36 \quad x - y + z = -9 \ (3)\\ \hline 7x + 2z &= -22 \quad (5) \end{aligned}$$

5.1 Example 6 Solving a System of Three Equations with Three Variables (cont.)

Solve the system consisting of equations (4) and (5) by multiplying equation (4) by –2 and then adding the result to equation (5) to solve for x.

$$\begin{aligned} -2x - 2z &= 12 \quad x + z = -6 \ (4)\\ 7x + 2z &= -22 \quad (5)\\ \hline 5x &= -10 \Rightarrow x = -2 \end{aligned}$$

Substitute –2 for x in equation (4) to solve for z.

$$-2 + z = -6 \Rightarrow z = -4$$

5.1 Example 6 Solving a System of Three Equations with Three Variables (cont.)

Substitute –2 for x and –2 for z in equation (3) to solve for y.

$$\begin{aligned} x - y + z &= -9 \quad (3)\\ -2 - y - 4 &= -9 \Rightarrow y = 3 \end{aligned}$$

$x = –2, y = 3, z = –4.$

Verify that (–2, 3, –4) satisfies all three equations in the original system.

$$\begin{aligned} 3x + 4y - 2z &= 14 \quad (1)\\ 3(-2) + 4(3) - 2(-4) &= 14 \quad ?\\ -6 + 12 + 8 &= 14\\ 14 &= 14 \quad \text{True} \end{aligned}$$

5.1 Example 6 Solving a System of Three Equations with Three Variables (cont.)

$$\begin{aligned} 2x + y + 2z &= -9 \quad (2)\\ 2(-2) + 3 + 2(-4) &= -9 \quad ?\\ -4 + 3 - 8 &= -9\\ -9 &= -9 \quad \text{True} \end{aligned}$$

$$\begin{aligned} x - y + z &= -9 \quad (3)\\ -2 - 3 + (-4) &= -9 \quad ?\\ -9 &= -9 \quad \text{True} \end{aligned}$$

Solution set:{(–2, 3, –4)}

5.1 Example 7 Solving a System of Two Equations with Three Variables (page 501)

Solve the system. Write the system with z arbitrary.

$$\begin{aligned} 3x + y - 2z &= -7 \quad (1)\\ 5x + 2y + z &= -6 \quad (2) \end{aligned}$$

Multiply equation (1) by –2, then add the result to equation (1) to eliminate y.

$$\begin{aligned} -6x - 2y + 4z &= 14\\ 5x + 2y + z &= -6\\ \hline -x + 5z &= 8 \quad (3) \end{aligned}$$

Now solve equation (3) for x.

$$-x + 5z = 8 \Rightarrow -x = -5z + 8 \Rightarrow x = 5z - 8$$

5.1 Example 7 Solving a System of Two Equations with Three Variables (cont.)

Substitute $5z – 8$ for x in equation (2), then solve for y.

$$\begin{aligned} 5(5z - 8) + 2y + z &= -6\\ 25z - 40 + 2y + z &= -6\\ 2y + 26z &= 34\\ 2y &= -26z + 34\\ y &= -13z + 17 \end{aligned}$$

With z arbitrary, the solution set is $\{(5z - 8, -13z + 17, z)\}$.

5.1 Example 8 Using Curve Fitting to Find an Equation Through Three Points (page 502)

Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points (–5, 7), (–1, –2), and (3, 5).

The three points must satisfy the equation. Substitute each ordered pair into the equation to obtain a system of equations.

$$7 = a(-5)^2 + b(-5) + c \Rightarrow 25a - 5b + c = 7 \quad (1)$$
$$-2 = a(-1)^2 + b(-1) + c \Rightarrow a - b + c = -2 \quad (2)$$
$$5 = a(3)^2 + b(3) + c \Rightarrow 9a + 3b + c = 5 \quad (3)$$

5.1 Example 8 Using Curve Fitting to Find an Equation Through Three Points (cont.)

Multiply equation (2) by –1, then add the result to equation (1) to eliminate *c*.

$$\begin{aligned} 25a - 5b + c &= 7 \quad (1) \\ -a + b - c &= 2 \quad a - b + c = 2 \quad (2) \\ \hline 24a - 4b &= 9 \quad (4) \end{aligned}$$

Multiply equation (2) by –1, then add the result to equation (3) to eliminate *c*.

$$\begin{aligned} -a + b - c &= 2 \quad a - b + c = 2 \quad (2) \\ 9a + 3b + c &= 5 \quad (3) \\ \hline 8a + 4b &= 7 \quad (5) \end{aligned}$$

5.1 Example 8 Using Curve Fitting to Find an Equation Through Three Points (cont.)

Add equations (4) and (5), then solve for *a*.

$$\begin{aligned} 24a - 4b &= 9 \quad (4) \\ 8a + 4b &= 7 \quad (5) \\ \hline 32a &= 16 \Rightarrow a = \tfrac{1}{2} \end{aligned}$$

Substitute ½ for *a* into equation (4) to solve for *b*.

$$24\left(\tfrac{1}{2}\right) - 4b = 9 \quad (4)$$
$$12 - 4b = 9 \Rightarrow -4b = -3 \Rightarrow b = \tfrac{3}{4}$$

5.1 Example 8 Using Curve Fitting to Find an Equation Through Three Points (cont.)

Now substitute ½ for *a* and ¾ for *b* into equation (2) to solve for *c*.

$$\tfrac{1}{2} - \tfrac{3}{4} + c = -2 \quad (2)$$
$$-\tfrac{1}{4} + c = -2 \Rightarrow c = -\tfrac{7}{4}$$

$$a = \tfrac{1}{2}, b = \tfrac{3}{4}, c = -\tfrac{7}{4}$$

The equation of the parabola is $y = \frac{1}{2}x^2 + \frac{3}{4}x - \frac{7}{4}$ or $y = .5x^2 + .75x - 1.75$.

5.1 Example 9 Solving an Application Using a System of Three Equations (page 503)

The table shows the number of units of protein, fat, and fiber are in one unit of each ingredient in an animal feed. How many units of each ingredient should be used to make a feed that contains 35 units of protein, 38 units of fat, and 28 units of fiber?

	Corn	Soybeans	Cottonseed	Total
Protein	.25	.4	.2	35
Fat	.4	.2	.3	38
Fiber	.3	.2	.1	28

5.1 Example 9 Solving an Application Using a System of Three Equations (cont.)

Let *x* = the number of units of corn.
Let *y* = the number of units of soybeans.
Let *z* = the number of units of cottonseed.

Since the total amount of protein is to be 35 units, the first row of the table yields $.25x + .4y + .2z = 35$ (1)

The total amount of fat is to be 38 units, so the second row of the table yields $.4x + .2y + .3z = 38$ (2)

Since the total amount of fiber is to be 28 units, the third row of the table yields $.3x + .2y + .1z = 28$ (3)

5.1 Example 9 Solving an Application Using a System of Three Equations (cont.)

$$\begin{aligned} .25x + .4y + .2z &= 35 \quad (1) \\ .4x + .2y + .3z &= 38 \quad (2) \\ .3x + .2y + .1z &= 28 \quad (3) \end{aligned}$$

Multiply both sides of equation (1) by 100 and both sides of equations (2) and (3) by 10 to obtain the equivalent system

$$\begin{aligned} 25x + 40y + 20z &= 3500 \quad (4) \\ 4x + 2y + 3z &= 380 \quad (5) \\ 3x + 2y + z &= 280 \quad (6) \end{aligned}$$

5.1 Example 9 Solving an Application Using a System of Three Equations (cont.)

Multiply equation (5) by –20, then add the result to equation (4) to eliminate y.

$$\begin{aligned} 25x + 40y + 20z &= 3500 \quad (4) \\ -80x - 40y - 60z &= -7600 \quad 4x + 2y + 3z = 380 \ (5) \\ \hline -55x \qquad - 40z &= -4100 \quad (7) \end{aligned}$$

Multiply equation (6) by –20, then add the result to equation (4) to eliminate y and z, then solve for x.

$$\begin{aligned} 25x + 40y + 20z &= 3500 \quad (4) \\ -60x - 40y - 20z &= -6600 \quad 3x + 2y + z = 280 \ (6) \\ \hline -35x \qquad &= -2100 \Rightarrow x = 60 \end{aligned}$$

5.1 Example 9 Solving an Application Using a System of Three Equations (cont.)

Substitute 60 for x in equation (7), then solve for z.

$$-55(60) - 40z = -4100 \quad -55x - 40z = -4100 \ (7)$$

$$-3300 - 40z = -4100 \Rightarrow -40z = -800 \Rightarrow z = 20$$

Substitute 60 for x and 20 for z in equation (6), then solve for y.

$$3(60) + 2y + 20 = 280 \quad 3x + 2y + z = 280 \ (6)$$

$$200 + 2y = 280 \Rightarrow 2y = 80 \Rightarrow y = 40$$

Verify that the ordered triple (60, 40, 20) satisfies the system formed by equations (1), (2), and (3).

The feed should contain 60 units of corn, 40 units of soybeans, and 20 units of cottonseed.

5.2 Matrix Solution of Linear Systems

The Gauss-Jordan Method ▪ Special Systems

5.2 Example 1 Using the Gauss-Jordan Method (page 512)

Solve the system.

$$\begin{aligned} 2x + 3y &= 7 \quad (1) \\ 3x - 4y &= -32 \quad (2) \end{aligned}$$

The system has the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 3 & -4 & -32 \end{array}\right]$$

Multiply each entry in the first row by ½ to get a 1 in the first row, first column position.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{7}{2} \\ 3 & -4 & -32 \end{array}\right] \frac{1}{2}\text{R1}$$

5.2 Example 1 Using the Gauss-Jordan Method (cont.)

Introduce 0 in the second row, first column by multiplying each element of the first row by –3 and then adding the result to the corresponding element of the second row.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & -\frac{17}{2} & -\frac{85}{2} \end{array}\right] -3\text{R1} + \text{R2}$$

Introduce 1 in the second row, second column by multiplying each element of the second row by $-\frac{2}{17}$.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 5 \end{array}\right] -\frac{2}{17}\text{R2}$$

5.2 Example 1 Using the Gauss-Jordan Method (cont.)

Finally, introduce 0 in the first row, second column by multiplying each element of the second row by $-\frac{3}{2}$ and then adding the result to the corresponding element of the first row.

$$\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 5 \end{array}\right] \quad -\tfrac{3}{2}R2 + R1$$

The last matrix corresponds to the system

$$x = -4$$
$$y = 5$$

Verify that (–4, 5) satisfies the original system.

Solution set: {(–4, 5)}

5.2 Example 2 Using the Gauss-Jordan Method (page 514)

Solve the system.

$$\begin{aligned} x + y - 4z &= 10 \quad (1) \\ 2x - 3y + z &= 7 \quad (2) \\ 3x - y - z &= 12 \quad (3) \end{aligned}$$

The system has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 2 & -3 & 1 & 7 \\ 3 & -1 & -1 & 12 \end{array}\right]$$

5.2 Example 2 Using the Gauss-Jordan Method (cont.)

There is already a 1 in the first row, first column.

Multiply each entry in the first row by –2, then add the result to the corresponding element of the second row to get a 0 in the second row, first column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & -5 & 9 & -13 \\ 3 & -1 & -1 & 12 \end{array}\right] \quad -2R1 + R2$$

5.2 Example 2 Using the Gauss-Jordan Method (cont.)

Multiply each entry in the second row by $-\frac{1}{5}$, to get a 1 in the second row, second column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & 1 & -\frac{9}{5} & \frac{13}{5} \\ 3 & -1 & -1 & 12 \end{array}\right] \quad -\tfrac{1}{5}R2$$

Multiply each entry in the first row by –3, then add the result to the corresponding element of the third row to get a 0 in the third row, first column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & 1 & -\frac{9}{5} & \frac{13}{5} \\ 0 & -4 & 11 & -18 \end{array}\right] \quad -3R1 + R3$$

5.2 Example 2 Using the Gauss-Jordan Method (cont.)

Multiply each entry in the second row by 4, then add the result to the corresponding element of the third row to get a 0 in the third row, second column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & 1 & -\frac{9}{5} & \frac{13}{5} \\ 0 & 0 & \frac{19}{5} & -\frac{38}{5} \end{array}\right] \quad 4R2 + R3$$

Multiply each entry in the third row by $\frac{5}{19}$, to get a 1 in the third row, third column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & 1 & -\frac{9}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \tfrac{5}{19}R3$$

5.2 Example 2 Using the Gauss-Jordan Method (cont.)

Multiply each entry in the third row by $\frac{9}{5}$, then add the result to the corresponding element of the second row to get a 0 in the second row, third column position.

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \tfrac{9}{5}R3 + R2$$

Subtract row 2 from row 1 to get a 1 in the first row, second column position.

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 11 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R1 - R2$$

5.2 **Example 2** Using the Gauss-Jordan Method (cont.)

Multiply each entry in the third row by 4, then add the result to the corresponding element of the first row to get a 0 in the first row, third column position.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R1+4R3 \\ \\ \\ \end{array}$$

Verify that (3, –1, –2) satisfies the original system.

Solution set: {(3, –1, –2)}

5.2 **Example 3** Solving an Inconsistent System (page 515)

Use the Gauss-Jordan method to solve the system.

$$\begin{aligned} 2x - 3y &= 7 \quad (1) \\ -6x + 9y &= 0 \quad (2) \end{aligned}$$

The system has the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -3 & 7 \\ -6 & 9 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 0 & 21 \\ -6 & 9 & 0 \end{array}\right] \begin{array}{l} 3R1+R2 \\ \\ \end{array}$$

The first row corresponds to the equation 0*x* + 0*y* = 21, which has no solution.

Solution set: ø

5.2 **Example 4** Solving a System with Infinitely Many Solutions (page 516)

Use the Gauss-Jordan method to solve the system. Write the solution set with *z* arbitrary.

$$\begin{aligned} 2x + y + z &= 5 \quad (1) \\ 3x + 2y - z &= -8 \quad (2) \end{aligned}$$

The system has the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 3 & 2 & -1 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 1 & -5 & -31 \end{array}\right] \begin{array}{l} \\ -3R1+2R2 \end{array}$$

5.2 **Example 4** Solving a System with Infinitely Many Solutions (cont.)

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 36 \\ 0 & 1 & -5 & -31 \end{array}\right] \begin{array}{l} R1+(-R2) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 18 \\ 0 & 1 & -5 & -31 \end{array}\right] \begin{array}{l} \frac{1}{2}R1 \\ \\ \end{array}$$

It is not possible to go further. The equations that correspond to the final matrix are

$$x + 3z = 18 \text{ and } y - 5z = -31.$$

Solve these equations for *x* and *y*, respectively

$$\begin{aligned} x + 3z = 18 &\Rightarrow x = -3z + 18 \\ y - 5z = -31 &\Rightarrow y = 5z - 31 \end{aligned}$$

Solution set: {(–3*z* + 18, 5*z* – 31, *z*)}

5.3 Determinant Solution of Linear Systems

Determinants ▪ Cofactors ▪ Evaluating *n* × *n* Determinants ▪ Cramer's Rule

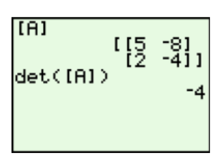
5.3 **Example 1** Evaluating a 2 × 2 Determinant (page 523)

Let $B = \begin{vmatrix} 5 & -8 \\ 2 & -4 \end{vmatrix}$. Find $|B|$.

$$\begin{vmatrix} 5 & -8 \\ 2 & -4 \end{vmatrix} = \underset{\uparrow\, a_{11}}{5} \cdot (\underset{\uparrow\, a_{22}}{-4}) - \underset{\uparrow\, a_{21}}{2} \cdot (\underset{\uparrow\, a_{12}}{-8}) = -20 + 16 = -4$$

a_{11} a_{12} a_{21} a_{22}

Graphing calculator solution



5.3 **Example 2** Finding Cofactors of Elements (page 525)

Find the cofactor of each of the following elements of the matrix.

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & 2 & -9 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) 5

Since 5 is in the first row, first column, $i = 1$ and $j = 1$.

$$M_{11} = \begin{vmatrix} 2 & -9 \\ 1 & 3 \end{vmatrix} = 15$$

The cofactor is $(-1)^{1+1}(15) = 15$.

5.3 **Example 2** Finding Cofactors of Elements (cont.)

(b) 0

Since 0 is in the second row, first column, $i = 2$ and $j = 1$.

$$M_{21} = \begin{vmatrix} -1 & 6 \\ 1 & 3 \end{vmatrix} = -9$$

The cofactor is $(-1)^{2+1}(-9) = 9$.

(c) 3

Since 3 is in the third row, third column, $i = 3$ and $j = 3$.

$$M_{33} = \begin{vmatrix} 5 & -1 \\ 0 & 2 \end{vmatrix} = 10$$

The cofactor is $(-1)^{3+3}(10) = 10$.

5.3 **Example 3** Evaluating a 3 × 3 Determinant (page 526)

Evaluate $\begin{vmatrix} 2 & -3 & 5 \\ 1 & 4 & -5 \\ 0 & 2 & -6 \end{vmatrix}$, expanding by the third row.

$$\begin{vmatrix} 2 & -3 & 5 \\ 1 & 4 & -5 \\ 0 & 2 & -6 \end{vmatrix} = a_{31} \cdot A_{31} + a_{32} \cdot A_{32} + a_{33} \cdot A_{33}.$$

It is not necessary to calculate A_{31} since $a_{31} = 0$.

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 1 & -5 \end{vmatrix} = -1(-15) = 15$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 11$$

5.3 **Example 3** Evaluating a 3 × 3 Determinant (page 526)

$$\begin{vmatrix} 2 & -3 & 5 \\ 1 & 4 & -5 \\ 0 & 2 & -6 \end{vmatrix} = a_{31} \cdot A_{31} + a_{32} \cdot A_{32} + a_{33} \cdot A_{33}.$$

$$= 0 \cdot A_{31} + 2 \cdot 15 - 6 \cdot 11 = -36$$

5.3 **Example 4** Applying Cramer's Rule to a 2 × 2 System (page 528)

Use Cramer's rule to solve the system.

$$4x + 3y = 2$$
$$x - 2y = 5$$

By Cramer's rule, $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$. Find D first, since if $D = 0$, Cramer's rule does not apply. If $D \neq 0$, find D_x and D_y.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 1 & -2 \end{vmatrix} = -11$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -19 \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} = 18$$

5.3 **Example 4** Applying Cramer's Rule to a 2 × 2 System (cont.)

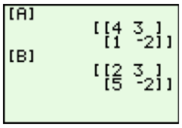
$$x = \frac{D_x}{D} = \frac{-19}{-11} = \frac{19}{11}$$

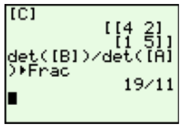
$$y = \frac{D_y}{D} = \frac{18}{-11} = -\frac{18}{11}$$

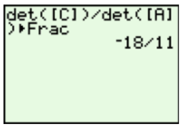
Solution set: $\left\{\left(\frac{19}{11}, -\frac{18}{11}\right)\right\}$

5.3 **Example 4** Applying Cramer's Rule to a 2 × 2 System
(cont.)

Graphing calculator solution







The screens support the algebraic solution.

5.3 **Example 4** Applying Cramer's Rule to a 3 × 3 System
(page 529)

Use Cramer's rule to solve the system.

$$\begin{aligned} 2x - 4y + z - 19 &= 0 \\ 4x + 6y - z + 15 &= 0 \\ x + y + 2z - 11 &= 0 \end{aligned}$$

Rewrite each equation in the form $ax + by + cz = k$.

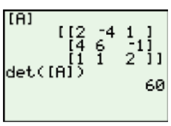
$$\begin{aligned} 2x - 4y + z &= 19 \\ 4x + 6y - z &= -15 \\ x + y + 2z &= 11 \end{aligned}$$

5.3 **Example 5** Applying Cramer's Rule to a 3 × 3 System
(cont.)

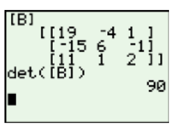
By Cramer's rule, $x = \frac{D_x}{D}, y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$. Find D first, since if $D = 0$, Cramer's rule does not apply.

Use a calculator to find the determinants.

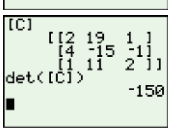
D



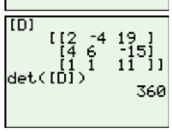
D_x



D_y



D_z



5.3 **Example 5** Applying Cramer's Rule to a 3 × 3 System
(cont.)

$$x = \frac{D_x}{D} = \frac{90}{60} = \frac{3}{2}$$

$$y = \frac{D_y}{D} = \frac{-150}{60} = -\frac{5}{2}$$

$$z = \frac{D_z}{D} = \frac{360}{60} = 6$$

Solution set: $\left\{\left(\frac{3}{2}, -\frac{5}{2}, 6\right)\right\}$

5.3 **Example 6** Showing That Cramer's Rule Does Not Apply
(page 530)

Show that Cramer's rule does not apply to the system.

$$\begin{aligned} x - 3y + 2z &= 5 \\ 3x + y - 4z &= 8 \\ -9x - 3y + 12z &= -24 \end{aligned}$$

We need to show that $D = 0$. Expanding about column 1 gives

$$D = \begin{vmatrix} 1 & -3 & 2 \\ 3 & 1 & -4 \\ -9 & -3 & 12 \end{vmatrix} = 1\begin{vmatrix} 1 & -4 \\ -3 & 12 \end{vmatrix} - 3\begin{vmatrix} -3 & 2 \\ -3 & 12 \end{vmatrix} - 9\begin{vmatrix} -3 & 2 \\ 1 & -4 \end{vmatrix}$$

$$= 0 - 3(-30) - 9(10) = 0$$

Thus, Cramer's rule does not apply to the system.

5.4 Partial Fractions

Decomposition of Rational Expression ▪ Distinct Linear Factors ▪ Repeated Linear Factors ▪ Distinct Linear and Quadratic Factors ▪ Repeated Quadratic Factors

5.4 Example 1 Finding a Partial Fraction Decomposition (page 536)

Find the partial fraction decomposition of

$$\frac{3x^3-3x^2+7x-4}{x^2-x}$$

The degree of the numerator is greater than the degree of the denominator, so first find the quotient.

$$\begin{array}{r} 3x \\ x^2-x\overline{)3x^3-3x^2+7x-4} \\ \underline{3x^3-3x^2} \\ 7x-4 \end{array}$$

5.4 Example 1 Finding a Partial Fraction Decomposition (cont.)

The quotient is $\frac{3x^3-3x^2+7x-4}{x^2-x}=3x+\frac{7x-4}{x^2-x}$.

Now find the partial fraction decomposition for $\frac{7x-4}{x^2-x}$.

Factor the denominator. $\frac{7x-4}{x^2-x}=\frac{7x-4}{x(x-1)}$

$=\frac{A}{x}+\frac{B}{x-1}$ Partial fraction decomposition

$7x-4=A(x-1)+Bx$ (1) Multiply by $x(x-1)$.

5.4 Example 1 Finding a Partial Fraction Decomposition (cont.)

Let $x = 1$. Then equation (1) becomes

$$7(1)-4=A(1-1)+B(1)\Rightarrow B=3$$

Let $x = 0$. Then equation (1) becomes

$$7(0)-4=A(0-1)+B(0)\Rightarrow -4=-A\Rightarrow A=4$$

$$\frac{3x^3-3x^2+7x-4}{x^2-x}=3+\frac{4}{x}+\frac{3}{x-1}$$

5.4 Example 2 Finding a Partial Fraction Decomposition (page 537)

Find the partial fraction decomposition of

$$\frac{x^2-4x+7}{(x-3)^3}$$

This is a proper fraction, and the denominator is already factored with repeated linear factors.

$$\frac{x^2-4x+7}{(x-3)^3}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}\Rightarrow$$

$$x^2-4x+7=A(x-3)^2+B(x-3)+C \quad (1)$$

5.4 Example 2 Finding a Partial Fraction Decomposition (cont.)

Substitute 3 for x in equation (1), then solve for C.

$$3^2-4(3)+7=A(3-3)^2+B(3-3)+C\Rightarrow C=4$$

We need to find a system of equations to find A and B.

Substitute 1 for x and 4 for C in equation (1):

$$1^2-4(1)+7=A(1-3)^2+B(1-3)+4\Rightarrow$$
$$4=4A-2B+4\Rightarrow 0=4A-2B \quad (2)$$

Substitute –1 for x and 4 for C in equation (1):

$$(-1)^2-4(-1)+7=A(-1-3)^2+B(-1-3)+4\Rightarrow$$
$$12=16A-4B+4\Rightarrow 8=16A-4B\Rightarrow 2=4A-B \quad (3)$$

5.4 Example 2 Finding a Partial Fraction Decomposition (cont.)

Solve the system of equations (2) and (3) to find A and B.

$$\begin{array}{l} 4A-2B=0 \quad (2) \\ \underline{4A-\ \ B=2 \quad (3)} \\ \quad\ -B=-2\Rightarrow B=2 \end{array}$$

$$4A-2=2 \ (3)\Rightarrow 4A=4\Rightarrow A=1$$

$$\frac{x^2-4x+7}{(x-3)^3}=\frac{1}{x-3}+\frac{2}{(x-3)^2}+\frac{4}{(x-3)^3}$$

5.4 Example 3 Finding a Partial Fraction Decomposition (page 537)

Find the partial fraction decomposition of

$$\frac{8x^2+10x-3}{(x+4)(x^2+1)}$$

The denominator has distinct linear and quadratic factors, and neither is repeated.

$$\frac{8x^2+10x-3}{(x+4)(x^2+1)}=\frac{A}{x+4}+\frac{Bx+C}{x^2+1}\Rightarrow$$

$$8x^2+10x-3=A(x^2+1)+(Bx+C)(x+4) \quad (1)$$

5.4 Example 3 Finding a Partial Fraction Decomposition (cont.)

Substitute –4 for x in equation (1), then solve for A.

$$8(-4)^2+10(-4)-3=A\left[(-4)^2+1\right]+\left[B(-4)+C\right]\left[(-4)+4\right]\Rightarrow$$

$$85=17A\Rightarrow A=5$$

Substitute 0 for x and 5 for A in equation (1):

$$8(0)^2+10(0)-3=5\left[(0)^2+1\right]+\left[B(0)+C\right](0+4)\Rightarrow$$

$$-3=5+4C\Rightarrow C=-2$$

5.4 Example 3 Finding a Partial Fraction Decomposition (cont.)

Substitute 1 for x, 5 for A, and -2 for C in equation (1), then solve for B.

$$8(1)^2+10(1)-3=5\left[(1)^2+1\right]+\left[B(1)-2\right](1+4)\Rightarrow$$

$$15=10+5B-10\Rightarrow B=3$$

$$\frac{8x^2+10x-3}{(x+4)(x^2+1)}=\frac{5}{x+4}+\frac{3x-2}{x^2+1}$$

5.4 Example 4 Finding a Partial Fraction Decomposition (page 539)

Find the partial fraction decomposition of

$$\frac{x^4+3x^3+4x^2+2x+4}{x(x^2+2)^2}$$

The denominator has a linear factor and repeated quadratic factors.

$$\frac{x^4+3x^3+4x^2+2x+4}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

5.4 Example 4 Finding a Partial Fraction Decomposition (cont.)

Multiply both sides by $x(x^2+2)^2$.

$$x^4+3x^3+4x^2+2x+4$$
$$=A(x^2+2)^2+(Bx+C)(x)(x^2+2)+(Dx+E)x \quad (1)$$

Substitute 0 for x in equation (1):

$$0^4+3(0)^3+4(0)^2+2(0)+4$$
$$=A(0^2+2)^2+(B(0)+C)(0)(0^2+2)+(D(0)+E)(0)\Rightarrow$$
$$4=4A\Rightarrow 1=A$$

5.4 Example 4 Finding a Partial Fraction Decomposition (cont.)

Substitute 1 for A in equation (1), then expand and combine terms on the right side.

$$x^4+3x^3+4x^2+2x+4$$
$$=(x^2+2)^2+(Bx+C)(x)(x^2+2)+(Dx+E)x$$
$$=x^4+4x^2+4+Bx^4+Cx^3+2Bx^2+2Cx+Dx^2+Ex$$
$$=(1+B)x^4+Cx^3+(4+2B+D)x^2+(2C+E)x+4 \quad (2)$$

To get additional equations involving the unknowns, equate the coefficients of like powers of x on the two sides of equation (2).

5.4 Example 4 Finding a Partial Fraction Decomposition (cont.)

$$1 = 1 + B \Rightarrow B = 0$$
$$3 = C$$
$$4 = 4 + 2B + D \quad (3)$$
$$2 = 2C + E \quad (4)$$

Substituting 0 for *B* in equation (3) gives *D* = 0.

Substituting 3 for *C* in equation (4) gives *E* = –4.

$$\frac{x^4 + 3x^3 + 4x^2 + 2x + 4}{x\left(x^2 + 2\right)^2} = \frac{1}{x} + \frac{3}{x^2 + 2} + \frac{-4}{\left(x^2 + 2\right)^2}$$

5.5 Nonlinear Systems of Equations

Solving Nonlinear Systems with Real Solutions ▪ Solving Nonlinear Systems with Nonreal Complex Solutions ▪ Applying Nonlinear Systems

5.5 Example 1 Solving a Nonlinear System by Substitution (page 543)

Solve the system.

$$x^2 + y = 6 \quad (1)$$
$$x - y = -4 \quad (2)$$

Solve equation (2) for *y*, then substitute that expression into equation (1) and solve for *x*:

$$x - y = -4 \Rightarrow y = x + 4$$
$$x^2 + (x + 4) = 6 \Rightarrow x^2 + x - 2 = 0 \Rightarrow$$
$$(x + 2)(x - 1) = 0 \Rightarrow x = -2 \text{ or } x = 1$$

5.5 Example 1 Solving a Nonlinear System by Substitution (cont.)

$$x^2 + y = 6 \quad (1)$$
$$x - y = -4 \quad (2)$$

Substituting –2 for *x* in equation (2) gives *y* = 2.

Substituting 1 for *x* in equation (2) gives *y* = 5.

Verify that the ordered pairs (–2, 2) and (1, 5) satisfy the original equations.

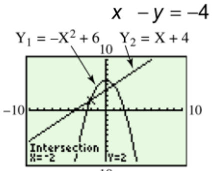
Solution set: {(–2, 2), (1, 5)}

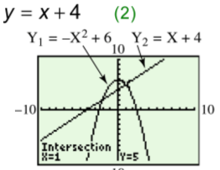
5.5 Example 1 Solving a Nonlinear System by Substitution (cont.)

Graphing calculator solution

Solve each equation for *y*, and graph them in the same viewing window.

$$x^2 + y = 6 \Rightarrow y = -x^2 + 6 \quad (1)$$
$$x - y = -4 \Rightarrow y = x + 4 \quad (2)$$





The points of intersection are (–2, 2), (1, 5).

5.5 Example 2 Solving a Nonlinear System by Elimination (page 544)

Solve the system.

$$x^2 + y^2 = 9 \quad (1)$$
$$9x^2 + 4y^2 = 36 \quad (2)$$

Multiply equation (1) by –4, then add the resulting equation (3) to equation (2).

$$-4x^2 - 4y^2 = -36 \quad (3)$$
$$9x^2 + 4y^2 = 36 \quad (2)$$
$$5x^2 = 0 \Rightarrow x = 0$$

5.5 **Example 2** Solving a Nonlinear System by Elimination (cont.)

Substitute 0 for x in equation (1), then solve for y.

$$x^2 + y^2 = 9 \quad (1)$$
$$y^2 = 9 \Rightarrow y = \pm 3$$

Verify that the ordered pairs (0, –3) and (0, 3) satisfy the original equations.

Solution set: {(0, –3) and (0, 3)}

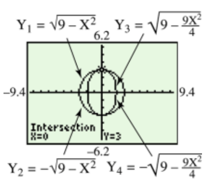
5.5 **Example 2** Solving a Nonlinear System by Elimination (cont.)

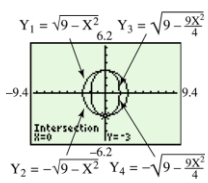
Graphing calculator solution

Solve each equation for y, and graph them in the same viewing window.

$$x^2 + y^2 = 9 \Rightarrow y = \pm\sqrt{9 - x^2} \quad (1)$$
$$9x^2 + 4y^2 = 36 \Rightarrow y = \pm\sqrt{9 - \frac{9x^2}{4}} = \pm 3\sqrt{1 - \frac{x^2}{4}} \quad (2)$$

5.5 **Example 2** Solving a Nonlinear System by Elimination (cont.)





The points of intersection are (0, –3) and (0, 3).

5.5 **Example 3** Solving a Nonlinear System by a Combination of Methods (page 545)

Solve the system.

$$x^2 + xy + y^2 = 21 \quad (1)$$
$$x^2 - xy + y^2 = 9 \quad (2)$$

Subtract equation (2) from equation (1) to obtain

$$2xy = 12 \Rightarrow xy = 6 \Rightarrow y = \frac{6}{x} \quad (3)$$

5.5 **Example 3** Solving a Nonlinear System by a Combination of Methods (cont.)

Substitute $\frac{6}{x}$ for y into equation (1) to solve for x.

$$x^2 + x\left(\frac{6}{x}\right) + \left(\frac{6}{x}\right)^2 = 21$$
$$x^2 + 6 + \frac{36}{x^2} = 21$$
$$x^2 - 15 + \frac{36}{x^2} = 0$$
$$x^4 - 15x^2 + 36 = 0$$
$$\left(x^2 - 3\right)\left(x^2 - 12\right) = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3} \text{ or}$$
$$x^2 = 12 \Rightarrow x = \pm 2\sqrt{3}$$

5.5 **Example 3** Solving a Nonlinear System by a Combination of Methods (cont.)

Substitute these values into equation (3) to solve for y.

$$x = \sqrt{3} \Rightarrow y = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$
$$x = -\sqrt{3} \Rightarrow y = -\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = -2\sqrt{3}$$
$$x = 2\sqrt{3} \Rightarrow y = \frac{6}{2\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$
$$x = -2\sqrt{3} \Rightarrow y = -\frac{6}{2\sqrt{3}} = -\frac{3\sqrt{3}}{3} = -\sqrt{3}$$

Solution set:

$$\left\{\left(\sqrt{3}, 2\sqrt{3}\right), \left(-\sqrt{3}, -2\sqrt{3}\right), \left(2\sqrt{3}, \sqrt{3}\right), \left(-2\sqrt{3}, -\sqrt{3}\right)\right\}$$

5.5 Example 4 Solving a Nonlinear System with an Absolute Value Equation (page 546)

Solve the system.

$$x^2 + y^2 = 4 \quad (1)$$
$$-|x| + y = 0 \quad (2)$$

Solving equation (2) for |x| gives |x| = y.

Since $|x| \geq 0$ for all x, $y \geq 0$.

In equation (1), the first term is x^2, which is the same as $|x|^2$, giving $y^2 + y^2 = 4 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$.

Reject the negative solution.

5.5 Example 4 Solving a Nonlinear System with an Absolute Value Equation (cont.)

Substitute $\sqrt{2}$ for y in equation (2) to solve for x.

$$-|x| + \sqrt{2} = 0 \Rightarrow -|x| = -\sqrt{2} \Rightarrow |x| = \sqrt{2} \Rightarrow$$
$$x = \sqrt{2} \text{ or } x = -\sqrt{2}$$

Verify that the ordered pairs $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$ satisfy the original system.

Solution set: $\{(-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \sqrt{2})\}$

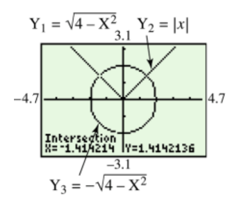
5.5 Example 4 Solving a Nonlinear System with an Absolute Value Equation (cont.)

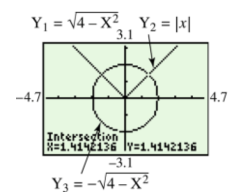
Graphing calculator solution

Solve each equation for y, and graph them in the same viewing window.

$$x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2} \quad (1)$$
$$-|x| + y = 0 \Rightarrow y = |x| \quad (2)$$

5.5 Example 4 Solving a Nonlinear System with an Absolute Value Equation (cont.)





The points of intersection are $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$.

5.5 Example 5 Solving a Nonlinear System with Nonreal Complex Numbers in its Solutions (page 547)

Solve the system.

$$x^2 + y^2 = 6 \quad (1)$$
$$3x^2 + 2y^2 = 8 \quad (2)$$

$$-3x^2 - 3y^2 = -18 \quad \text{Multiply (1) by } -3.$$
$$3x^2 + 2y^2 = 8 \quad (2)$$
$$-y^2 = -10 \Rightarrow y = 10 \Rightarrow y = \pm\sqrt{10}$$

5.5 Example 5 Solving a Nonlinear System with Nonreal Complex Numbers in its Solutions (cont.)

Find the corresponding values of x by substituting the values of y into equation (1).

$$y = \sqrt{10} \Rightarrow x^2 + (\sqrt{10})^2 = 6 \Rightarrow x^2 = -4 \Rightarrow x = \pm 2i$$
$$y = -\sqrt{10} \Rightarrow x^2 + (-\sqrt{10})^2 = 6 \Rightarrow x^2 = -4 \Rightarrow x = \pm 2i$$

Verify that the ordered pairs $(2i, \sqrt{10})$, $(-2i, \sqrt{10})$, $(2i, -\sqrt{10})$, and $(-2i, -\sqrt{10})$ satisfy the original system.

Solution set:
$\{(2i, \sqrt{10}), (-2i, \sqrt{10}), (2i, -\sqrt{10}), (-2i, -\sqrt{10})\}$

5.5 Example 5 Using a Nonlinear System to Find the Dimensions of a Box (page 547)

The length of the hypotenuse of a right triangle is 41 cm. One of the legs is 31 cm longer than the other. Find the lengths of the two legs of the triangle.

Let x = the length of one leg
Let y = the length of the other leg.
Then, we have the system:

$$x^2 + y^2 = 41^2 \quad (1) \quad \text{Pythagorean theorem}$$
$$x - y = 31 \quad (2)$$

5.5 Example 5 Using a Nonlinear System to Find the Dimensions of a Box (cont.)

Solve equation (2) for x, then substitute that expression into equation (1) to solve for y.

$$x = y + 31 \quad \text{From equation (2)}$$
$$(y+31)^2 + y^2 = 41^2 \Rightarrow 2y^2 + 62y - 720 = 0 \Rightarrow$$
$$y^2 + 31y - 360 = 0 \Rightarrow (y-9)(y+40) = 0 \Rightarrow$$
$$y = 9 \quad \text{or} \quad y = -40$$

Reject the negative solution since length cannot be negative.

Substitute $y = 9$ into equation (2) to solve for x.

$$x - 9 = 31 \Rightarrow x = 40$$

The lengths of the legs are 40 cm and 9 cm.