Chapter 6: Continuous Random Variables & the Normal Distribution

- Continuous random variables & their distributions
- Normal distributions
  - Standard normal distribution
  - Standardizing a normal distribution
- Related problems
  - Find probabilities given points of a normal distribution
  - Find points given probability of a normal distribution
- Normal approximation to the Binomial distribution

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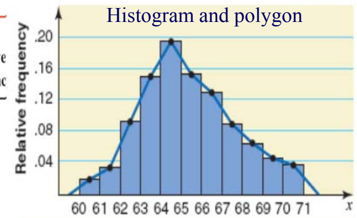
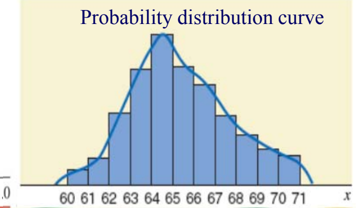
6.1 Continuous Probability Distribution

Height of a Female Student (inches) x	f	Relative Frequenc
60 to less than 61	90	.018
61 to less than 62	170	.034
62 to less than 63	460	.092
63 to less than 64	750	.150
64 to less than 65	970	.194
65 to less than 66	760	.152
66 to less than 67	640	.128
67 to less than 68	440	.088
68 to less than 69	320	.064
69 to less than 70	220	.044
70 to less than 71	180	.036
	$N = 5000$	Sum = 1.0





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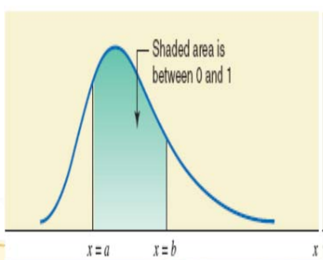
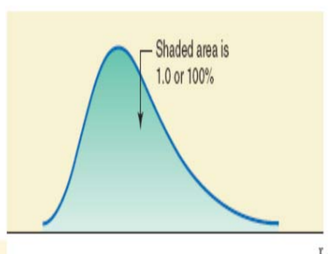
Two Characteristics of Continuous Distributions

- The probability that x assumes a value in any interval lies in the range 0 to 1
  - The probability of a single value of x (x=a, x=b & a=b) is **zero**
- The total probability of all the (mutually exclusive) intervals within which x can assume a value of 1.0





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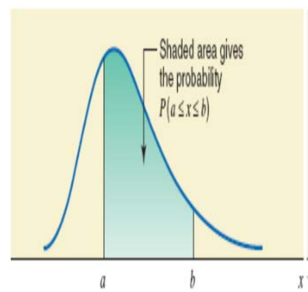
Area under the Curve as Probability



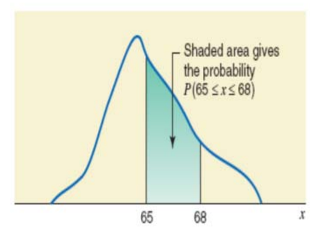


Figure 6.6 Probability that x lies in the interval 65 to 68.

⇧

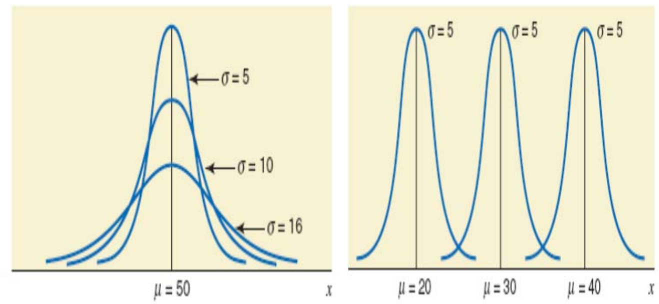
Probability that x lies in the interval 65 to 68 inches

6.2 The Normal Distribution

- A special continuous probability distribution
  - Bell – shaped curve
    - Two parameters μ & σ
  - Symmetric about the mean
  - Mean μ determines its central location
  - Standard deviation σ determines its shape
    - Large σ gives flatter bell-curve
    - small σ gives steeper bell-curve
  - Two tails of the curve extend indefinitely
- Total area (probability) = 1
  - Symmetry is often used to figure out probability
  - Areas of the normal curve beyond $\mu \pm 3\sigma$; recall empirical rule
- How to find probability (area)?

Normal Distribution Curves

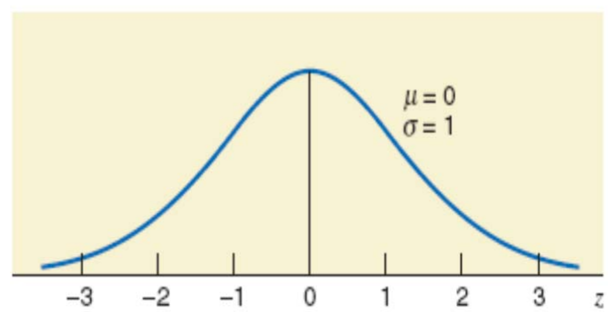
- Mean μ determines its central location; standard deviation σ determines its shape



6.3 The Standard Normal Distribution

- There are so many general normal distributions with different mean μ and standard deviation σ
- We focus one here – standard normal distribution with $\mu = 0$ & $\sigma = 1$
  - Other general normal distributions can be standardized into the standard normal distribution
  - Reserve **Z** for the random variable having the standard normal distribution
  - z values or z scores – the units or values marked on the horizontal axis of the standard normal curve. A specific value of z gives the distance between the mean and the point represented by z in terms of the standard deviation (since $\sigma = 1$)
- How to find the probability?
  - Integral of calculus, table, Excel, TI – 83, …

The Standard Normal Distribution Curve



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How to Find the Probability – Table IV

- Table IV gives the Area under the Standard Normal Curve to the Left of z score – left tail from z, or cumulative probability
  - Structure and limitation of the table

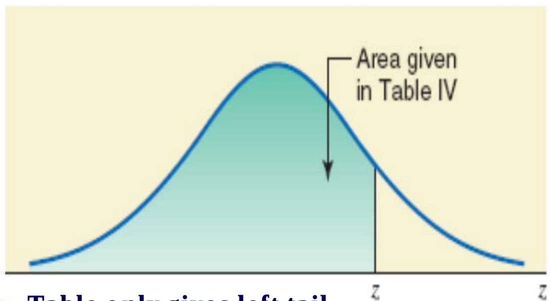

Table only gives left tail

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Example 6-1

- Find the area under the standard normal curve to the left of z = 1.95

z	.00	.01	. . .	.05	. . .	.09
−3.4	.0003	.0003	. . .	.0003	. . .	.0002
−3.3	.0005	.0005	. . .	.0004	. . .	.0003
−3.2	.0007	.0007	. . .	.0006	. . .	.0005
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
1.9	.9713	.9719	. . .	.9744	. . .	.9767
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
3.4	.9997	.9997	. . .	.9997	. . .	.9998

Structure and limitation of the table Required area

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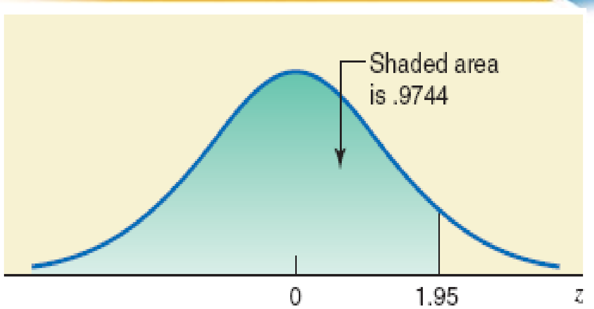
Solution to Example 6.1



Left tail from the table directly

- Solution to Example 6.1
  - P(Z <1.95) = "left tail" = 0.9744

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Example 6-2

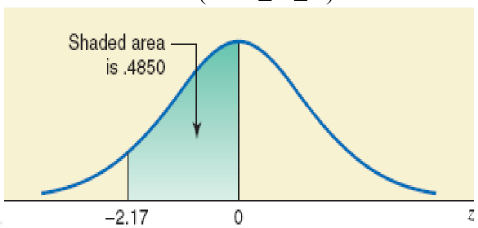
- Find the area under the standard normal curve from z = -2.17 to z = 0

z	.00	.01	. . .	.07	. . .	.09
−3.4	.0003	.0003	. . .	.0003	. . .	.0002
−3.3	.0005	.0005	. . .	.0004	. . .	.0003
−3.2	.0007	.0007	. . .	.0005	. . .	.0005
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
−2.1	.0179	.0174	. . .	.0150	. . .	.0143
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
0.0	.5000	.5040	. . .	.5279	. . .	.5359
.	.	.	. . .	.	. . .	.
.	.	.	. . .	.	. . .	.
3.4	.9997	.9997	. . .	.9997	. . .	.9998

Area to the left of $z = 0$ Area to the left of $z = -2.17$

Example 6-2: Solution

- To find the area from z=-2.17 to z =0 , first we find the areas to the left of z=0 and to the left of z=-2.17 in Table IV. These two areas are .5 and .0150, respectively. Next we subtract .0150 from .5 to find the required area.
- Area from -2.17 to 0 = $P(-2.17 \le z \le 0)$ = .5000 - .0150 = **.4850**

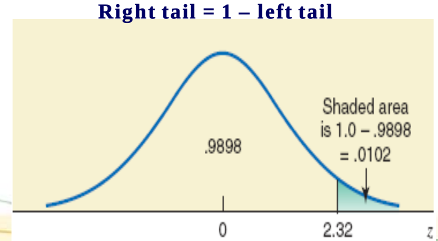


Middle Area = big left tail – small left tail

Example 6-3

- Find the area under the standard normal curve to the right of z = 2.32
- Solution
  - To find the area to the right of z=2.32, first we find the area to the left of z=2.32. Then we subtract this area from 1.0, which is the total area under the curve. The required area is 1.0 - .9898 = .0102.

Right tail = 1 – left tail



More Examples

- Steps to find probability
  - Sketch the normal curve and shade the area of probability. This first step is critical for most people
  - Change the problem into left tail problem
    - Left tail = table
    - Right tail = 1 – left tail
    - Middle area = big left tail – small left tail
- Find the following probabilities for the standard normal curve
  - P (1.19 < z < 2.12)
  - P (-1.56 < z < 2.31)
  - P (z > -.75)
  - P (0 < z < 5.67)
  - P (z < -5.35)
  - Empirical rules

Calculator TI-83: 2nd => DISTR
normalcdf(a,b) gives P(a < z < b)

Excel: norm.**S**.dist(a) gives P(z ≤ a)

6.4 Standardize a Normal Distribution

- We just discussed how to find probability under standard normal distribution. Now how to find probability under general normal distribution?
  - We use **X** for general normal random variable
- The answer is **through standardization**.
  - Converting an x value into a z score where μ and σ are the mean and standard deviation of the normal distribution of x, respectively $$z = \frac{x-\mu}{\sigma}$$
  - Standardization changes the shape of normal distribution, but it does not change the area.
- In general, $P(x_1 < \mathbf{X} < x_2) = P(z_1 < \mathbf{Z} < z_2)$, where $z_i = (x_i - \mu)/\sigma$, $i = 1, 2$

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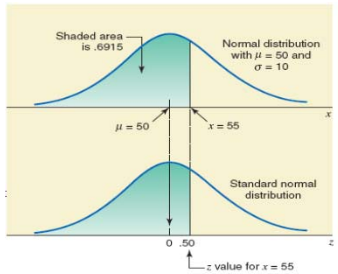
Example 6-6

- Let *x* be a continuous random variable that has a normal distribution with a mean of 50 and a standard deviation of 10. Convert the following *x* values to *z* values and find the probability to the left of these points
  - x = 55
  - x = 35
- Solution
  - x = 55
    - P(x < 55) = P(*z* < .50) = .6915

$$z = \frac{x-\mu}{\sigma} = \frac{55-50}{10}$$

  - x = 35
    - P(x < 35) = P(*z* < -1.50) = .0668

$$z = \frac{x-\mu}{\sigma} = \frac{35-50}{10} = -1.50$$



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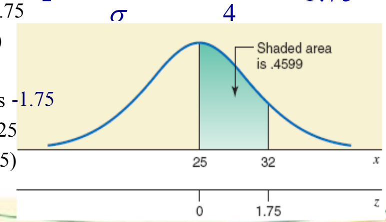
Example 6-7

- Let *x* be a continuous random variable that is normally distributed with a mean of 25 and a standard deviation of 4. Find the area (a) between x = 25 and x = 32 &
  (b) between x = 18 and x = 34
- Solution
  - (a) the z value for x = 25 is 0
    The z value for x = 32 is 1.75
    P (25<x<32)=P(0<z<1.75)
    = .4599
  - (b) the z value for x = 18 is -1.75
    the z value for x = 34 is 2.25

  P (18<x<34)=P (-1.75<z<2.25)
  = .9878 - .0401 = .9477

$$z = \frac{x-\mu}{\sigma} = \frac{32-25}{4} = 1.75$$



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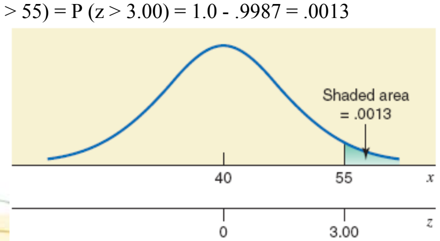
Example 6-8

- Let x be a normal random variable with its mean equal to 40 and standard deviation equal to 5. Find the probability of to the right of x = 55, i.e., P (x > 55)
- Solution
  - For x = 55: $z = \frac{55-40}{5} = 3.00$
  - P (x > 55) = P (z > 3.00) = 1.0 - .9987 = .0013



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More Examples

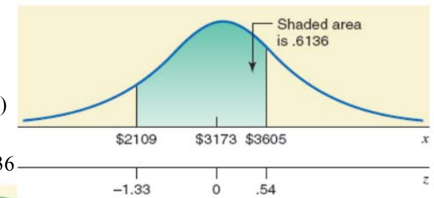
- Steps to find probability
  - Sketch the normal curve and shade the area of probability.
  - Standardize the general normal into standard normal and then follow the methods discussed in 6.3
  - Other methods: Excel & TI – 83 & 84
- Let *x* be a continuous random variable that has a normal distribution with μ = 50 and σ = 8. Find the probability *P* (30 ≤ *x* ≤ 39)
- Let x be a continuous random variable that has a normal distribution with a mean of 80 and a standard deviation of 12. Find the area under the normal distribution curve from (a) x = 70 to x = 135; (b) to the left of 27

6.5 Applications of the Normal Distribution

- Word problems
  - First identify mean μ and standard deviation σ
  - Translate words or sentences into probability
- Apply exact same idea of 6.4
  - Standardization
  - Finding probability
- Let's see a few examples

Example 6-11

- According to a Sallie Mae and credit bureau data, in 2008, college students carried an average of \$3173 debt on their credit cards (*USA TODAY*, April 13, 2009). Suppose that current credit card debts for all college students have a normal distribution with a mean of \$3173 and a standard deviation of \$800. Find the probability that credit card debt for a randomly selected college student is between \$2109 and \$3605.
- Solution
  - $\mu = 3173$ & $\sigma = 800$
  - P (\$2109 < x < \$3605)
    = P (-1.33 < z < .54)
    = .7054 - .0918 = .6136
    = 61.36%



Example 6-12 through 6-14

- Example 6 – 12
  - $\mu = 55$ & $\sigma = 4$
  - $P(x \le 60) = P(z \le 1.25) = .8944$
- Example 6 – 13
  - $\mu = 12$ & $\sigma = 0.015$
  - (a) P (11.97 ≤ x ≤ 11.99) = P (-2.00 ≤ z ≤ -.67) = .2514 - .0228 = .2286
  - *(b) P (12.02 ≤ x ≤ 12.07) = P (1.33 ≤ z ≤ 4.67) = 1 - .9082 = .0918*
- Example 6 – 14
  - $\mu = 54$ & $\sigma = 8$
  - $P(x < 36) = P(z < -2.25) = .0122$

6.6 Determining the z & x Values When an Area Under the Normal Distribution Curve is Known

- We have discussed that
  - given a z score, we find a unique probability (left tail)
  - given an x value, we find a unique probability (left tail)
- Now we go backwards or inversely.
  - given a probability of standard normal distribution **Z**, find the corresponding z score
  - given a probability of general normal distribution **X**, find the corresponding x value
- Key points
  - Find left tail from the z (x) point based on the given probability
  - then use Table IV, calculator TI-83, or Excel to find the z (x) values

Example 6-15

- Find a point *z* such that the area under the standard normal curve to the left of *z* is .9251.
  - Left tail of z is given directly
- Solution

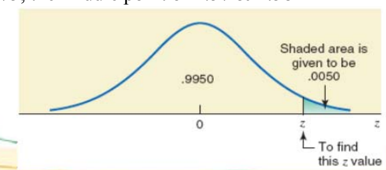
z = 1.44

z	.00	.01	⋯	.04	⋯	.09
−3.4	.0003	.0003	⋯		⋯	.0002
−3.3	.0005	.0005	⋯		⋯	.0003
−3.2	.0007	.0007	⋯		⋯	.0005
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
1.4				.9251	⋯	⋯
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
3.4	.9997	.9997	⋯	.9997	⋯	.9998

We locate this value in Table IV of Appendix C

Example 6-16

- Find the value of *z* such that the area under the standard normal curve in the right tail is .0050
- Solution
  - Right tail = 0.005. So the left tail of z is 1 – 0.0050 = 0.9950. The table does not contain 0.9950
    - Find the value closest to .9950, which is either .9949 or .9951.
    - If we choose .9951, the z = 2.58.
    - If we choose .9949, the z = 2.57.
  - z = 2.575, the middle point of 2.57 & 2.58



More Examples

- Steps to find z, given probability
  - Find the left tail based on given probability
  - Use the left tail to check the Table, or use TI-83 or Excel to find z
    - **TI – 83**: 2nd => DISTR => invNorm(**left-tail**)
    - **Excel**: norm.S.inv(**left-tail**)
- Find z such that
  - $P(Z \le z) = 0.95$
  - $P(Z > z) = 0.05$
  - $P(Z \le z) = 0.05$
  - $P(-z \le Z \le z) = 0.90$
  - $P(-z \le Z \le z) = 0.95$
  - $P(-z \le Z \le z) = 0.99$

Finding an *x* Value for a Normal Distribution

- For a normal curve, with known values of μ and σ and for a given area under the curve to the left of *x*, the *x* value is calculated as $x = \mu + z\sigma$
- Three Steps to Find an x value
  - Find the left tail of x based on given probability
  - Find ***z*** score of the left tail obtained
  - Find x value based on the formula $x = \mu + z\sigma$

Example 6-18

- Recall Example 6-14. It is known that the life of a calculator manufactured by Texas Instruments has a normal distribution with a mean of 54 months and a standard deviation of 8 months. What should the warranty period be to replace a malfunctioning calculator if the company does not want to replace more than 1% of all the calculators sold?
- Solution
  - Area to the left of x = .01 or 1%
  - Find the z value from the normal distribution table for .0100. Table IV does not contain a value that is exactly .0100.
    The value closest to .0100 in the table is .0099. The z = -2.33.
  - $x = \mu + z\sigma = 54 + (-2.33)(8) = 54 - 18.64 = 35.36$
  - Thus, the company should replace all calculators that start to malfunction within 35 months

Example 6-19

- Almost all high school students who intend to go to college take the SAT test. In a recent test, the mean SAT score (in verbal and mathematics) of all students was 1020. Debbie is planning to take this test soon. Suppose the SAT scores of all students who take this test with Debbie will have a normal distribution with a mean of 1020 and a standard deviation of 153. What should her score be on this test so that only 10% of all examinees score higher than she does?
- Solution
  - Area to the left of the x value = 1.0 - .10 = .9000
  - Look for .9000 in the body of the normal distribution table. The value closest to .9000 in Table IV is .8997, and the z value is 1.28.
  - $x = \mu + z\sigma = 1020 + 1.28(153) = 1020 + 195.84 = 1215.84 \approx 1216$
  - Thus, if Debbie scores 1216 on the SAT, only about 10% of the examinees are expected to score higher than she does.

6.7 Normal Approximation to Binomial Distribution

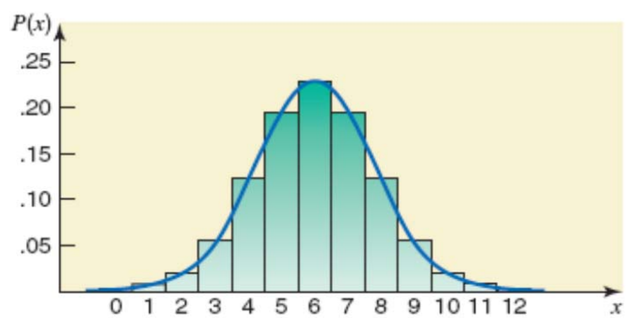


Figure 6.50 Histogram for the probability distribution of Table 6.5.

Summary

- Introduction to standard normal and general normal distributions and their properties
- Find probability based on given z scores or x values
  - Left tail (probability) is obtained directly from the Table, Excel
  - Right tail = 1 – left tail
  - Middle area = big left tail – small left tail
- Find z scores or x values based on given probability
  - Calculate the left tail based on the probability
  - Find the z score based on the obtained left tail by using the Table, Excel & TI-83
  - Find the x value if the distribution is not standard normal distribution by using formula $\boldsymbol{x = \mu + z\sigma}$