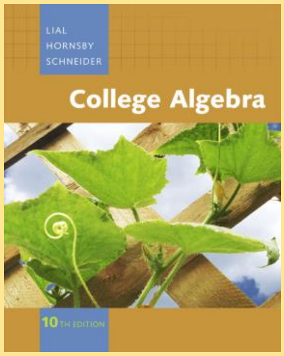
3

Polynomial and Rational Functions

Sections 3.1–3.4



3 Polynomial and Rational Functions

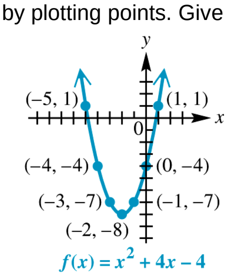
3.1 Quadratic Functions and Models

Quadratic Functions ▪ Graphing Techniques ▪ Completing the Square ▪ The Vertex Formula ▪ Quadratic Models and Curve Fitting

3.1 Example 1(a) Graphing Quadratic Functions (page 304)

Graph $f(x) = x^2 + 4x - 4$ by plotting points. Give the domain and range.

x	$f(x) = x^2 + 4x - 4$
–5	1
–4	–4
–3	–7
–2	–8
–1	–7
0	–4
1	1

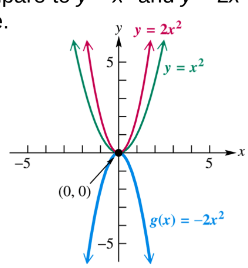


Domain: $(-\infty, \infty)$ Range: $[-8, \infty)$

3.1 Example 1(b) Graphing Quadratic Functions (page 304)

Graph $g(x) = -2x^2$ and compare to $y = x^2$ and $y = 2x^2$. Give the domain and range.

The graph of $g(x) = -2x^2$ is a narrower version of the graph of $y = x^2$ and is a reflection of the graph of $y = 2x^2$ across the x-axis.

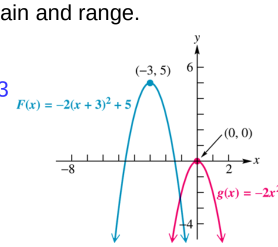


Domain: $(-\infty, \infty)$ Range: $(-\infty, 0]$

3.1 Example 1(c) Graphing Quadratic Functions (page 304)

Graph $F(x) = -2(x + 3)^2 + 5$ and compare to the graph in part (b). Give the domain and range.

The graph of $F(x)$ is the graph of $g(x)$ translated 3 units to the left and 5 units up.



Domain: $(-\infty, \infty)$ Range: $(-\infty, 5]$

3.1 **Example 2** Graphing a Parabola by Completing the Square (page 305)

Graph $f(x) = x^2 + 2x - 5$ by completing the square and locating the vertex.

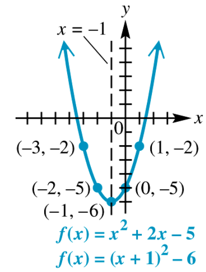
$$f(x) = (x^2 + 2x + \quad) - 5 \qquad \text{Complete the square.}$$
$$f(x) = (x^2 + 2x + 1 - 1) - 5 \qquad \text{Add and subtract 1.}$$
$$f(x) = (x^2 + 2x + 1) - 1 - 5 \qquad \text{Regroup terms.}$$
$$f(x) = (x + 1)^2 - 6 \qquad \text{Factor and simplify.}$$

Vertex (–1, –6) Axis: $x = -1$

3.1 **Example 2** Graphing a Parabola by Completing the Square (cont.)

Now find additional ordered pairs that satisfy the equation $f(x) = (x + 1)^2 - 6$.

x	$f(x) = (x+1)^2 - 6$
–3	–2
–2	–5
0	–5
1	–2



3.1 **Example 3** Graphing a Parabola by Completing the Square (page 305)

Graph $f(x) = 2x^2 + x - 6$ by completing the square and locating the vertex.

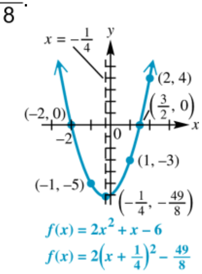
$$f(x) = 2\left(x^2 + \frac{1}{2}x + \quad\right) - 6 \qquad \text{Factor 2 from the first two terms.}$$
$$f(x) = 2\left(x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}\right) - 6 \qquad \text{Add and subtract } \left[\frac{1}{2}\left(\frac{1}{2}\right)\right]^2 = \frac{1}{16}.$$
$$f(x) = 2\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + 2\left(-\frac{1}{16}\right) - 6 \qquad \text{Distributive property}$$
$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{49}{8} \qquad \text{Factor and simplify.}$$

Vertex: $\left(-\frac{1}{4}, -\frac{49}{8}\right)$ Axis: $x = -\frac{1}{4}$

3.1 **Example 3** Graphing a Parabola by Completing the Square (cont.)

Now find additional ordered pairs that satisfy the equation $f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{49}{8}$.

x	$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{49}{8}$
–2	0
–1	–5
0	–6
1	–3
$\frac{3}{2}$	0
2	4



3.1 **Example 4** Finding the Axis and the Vertex of a Parabola Using the Vertex Formula (page 308)

Find the axis and vertex of the parabola $f(x) = -3x^2 + 12x - 8$ using the vertex formula.

$a = -3, b = 12, c = -8$

$$x = h = -\frac{b}{2a} = -\frac{12}{2(-3)} = 2$$

Axis: $x = 2$

Vertex: (2, f(2))

$$f(2) = -3(2)^2 + 12(2) - 8 = 4$$

Vertex: (2, 4)

3.1 **Example 5(a)** Solving a Problem Involving Projectile Motion (page 308)

A ball is thrown directly upward from an initial height of 75 ft with an initial velocity of 112 ft per sec. Give the function that describes the height of the ball in terms of time *t.*

The projectile height function is $s(t) = -16t^2 + v_0 t + s_0$.

$v_0 = 112$ and $s_0 = 75$, so

$$s(t) = -16t^2 + 112t + 75$$

3.1 Example 5(b) Solving a Problem Involving Projectile Motion (page 308)

After how many seconds does the ball reach its maximum height? What is the maximum height?

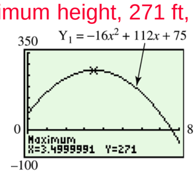
The maximum height occurs at the vertex.

$$a = -16, b = 112 \Rightarrow x = -\frac{b}{2a} = -\frac{112}{2(-16)} = 3.5$$

$$y = -16(3.5)^2 + 112(3.5) + 75 = 271$$

The ball reaches its maximum height, 271 ft, after 3.5 seconds.

Verify with a graphing calculator.



3.1 Example 5(c) Solving a Problem Involving Projectile Motion (page 308)

For what interval of time is the height of the ball greater than 200 ft?

Solve the quadratic inequality $-16t^2 + 112t + 75 > 200$.

$$-16t^2 + 112t + 75 > 200 \Rightarrow -16t^2 + 112t - 125 > 0$$

Use the quadratic formula to find the values of x that satisfy $-16t^2 + 112t - 125 = 0$.

$a = -16$, $b = 112$, $c = -125$

$$t = \frac{-112 \pm \sqrt{112^2 - 4(-16)(-125)}}{2(-16)} = \frac{-112 \pm \sqrt{4544}}{-32}$$

$$t = \frac{-112 + \sqrt{4544}}{-32} \approx 1.39 \quad \text{or} \quad t = \frac{-112 - \sqrt{4544}}{-32} \approx 5.61$$

3.1 Example 5(c) Solving a Problem Involving Projectile Motion (cont.)

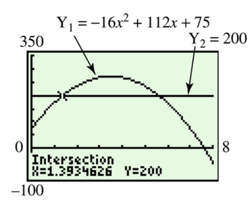
The two numbers divide a number line into three regions, $(-\infty, 1.39)$, $(1.39, 5.61)$, and $(5.61, \infty)$. Choose test values to see which interval satisfies the inequality.

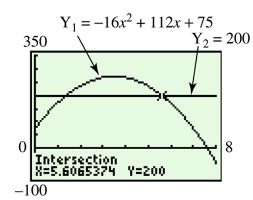
Interval	Test Value	Is $-16t^2 + 112t - 125 > 0$ True or False?
$(-\infty, 1.39)$	0	$-16 \cdot 0^2 + 112 \cdot 0 - 125 \overset{?}{>} 0$ $-125 > 0$ False
$(1.39, 5.61)$	2	$-16 \cdot 2^2 + 112 \cdot 2 - 125 \overset{?}{>} 0$ $35 > 0$ True
$(5.61, \infty)$	10	$-16 \cdot 10^2 + 112 \cdot 10 - 125 \overset{?}{>} 0$ $-605 > 0$ False

The ball will be greater than 200 ft above ground level between 1.39 and 5.61 seconds after it is thrown.

3.1 Example 5(c) Solving a Problem Involving Projectile Motion (cont.)

Verify with a graphing calculator.





3.1 Example 5(d) Solving a Problem Involving Projectile Motion (page 308)

After how many seconds will the ball hit the ground?

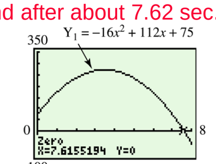
Use the quadratic formula to find the positive solution of $-16t^2 + 112t + 75 = 0$.

$$t = \frac{-112 \pm \sqrt{112^2 - (4)(-16)(75)}}{2(-16)} \approx \cancel{-0.62} \text{ or } 7.62$$

Reject

The ball hits the ground after about 7.62 sec.

Verify with a graphing calculator.



3.1 Example 6(a) Modeling the Number of Hospital Outpatient Visits (page 310)

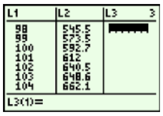
The table shows the number of hospital visits (in millions) for selected years. In the table, 80 represents 1980, 100 represents 2000, etc.

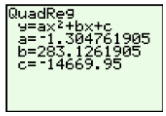
Year	Visits	Year	Visits
80	263.0	99	573.5
90	368.2	100	592.7
95	483.2	101	612.0
96	505.5	102	640.5
97	520.6	103	648.6
98	545.5	104	662.1

Source: American Hospital Association, U.S. Census Bureau.

Determine a quadratic model for the data for hospital outpatient visits for the years 1998–2004.

3.1 **Example 6(a)** Modeling the Number of Hospital Outpatient Visits (cont.)





Using the data for the years 1998−2004 and the quadratic regression function on the graphing calculator, we have $y = -1.3048x^2 + 283.13x - 14{,}670$.

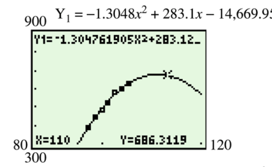
3.1 **Example 6(b)** Modeling the Number of Hospital Outpatient Visits (page 310)

Use the model from part (a) to predict the number of visits in 2010.

2010 corresponds to $x = 110$.

$$f(110) \approx -1.3048\left(110^2\right) + 283.13(110) - 14{,}670$$
$$\approx 686 \text{ million}$$

Verify with a graphing calculator.



3.2 Synthetic Division

Synthetic Division ▪ Evaluating Polynomial Functions Using the Remainder Theorem ▪ Testing Potential Zeros

3.2 **Example 1** Using Synthetic Division (page 323)

Use synthetic division to divide $\dfrac{4x^3 - 15x^2 + 11x - 10}{x - 3}$.

$x - 3$ is in the form $x - k$.

$$3\overline{)4 \quad -15 \quad 11 \quad -10} \leftarrow \text{Coefficients}$$

Bring down the 4, and multiply 3(4) = 12.

$$\begin{array}{r|rrrr} 3 & 4 & -15 & 11 & -10 \\ & \downarrow & 12 & & \\ \hline & 4 & & & \end{array}$$

3.2 **Example 1** Using Synthetic Division (cont.)

Add –15 and 12 to obtain –3. Multiply (–3)(3) = –9.

$$\begin{array}{r|rrrr} 3 & 4 & -15 & 11 & -10 \\ & & 12 & -9 & \\ \hline & 4 & -3 & & \end{array}$$

Add 11 and –9 to obtain 2. Multiply (2)(3) = 6.

$$\begin{array}{r|rrrr} 3 & 4 & -15 & 11 & -10 \\ & & 12 & -9 & 6 \\ \hline & 4 & -3 & 2 & \end{array}$$

3.2 **Example 1** Using Synthetic Division (cont.)

Add –10 and 6 to obtain 4.

$$\begin{array}{r|rrrr} 3 & 4 & -15 & 11 & -10 \\ & & 12 & -9 & 6 \\ \hline & \underbrace{4 \quad -3 \quad 2}_{\text{Quotient}} & & & -4 \leftarrow \text{Remainder} \end{array}$$

Since the divisor $x - k$ has degree 1, the degree of the quotient will always be one less than the degree of the original polynomial.

$$\frac{4x^3 - 15x^2 + 11x - 10}{x - 3} = 4x^2 - 3x + 2 + \frac{-4}{x - 3}$$

3.2 Example 2 Applying the Remainder Theorem (page 324)

Let $f(x) = -3x^4 + 15x^3 - 50x + 25$. Use the remainder theorem to find $f(4)$.

Use synthetic division with $k = 4$.

4)	−3	15	0	−50	25	
		−12	12	48	−8	
	−3	3	12	−2	17	← Remainder

$f(4) = 17$

3.2 Example 3(a) Deciding Whether a Number is a Zero (page 325)

Let $f(x) = 2x^3 - 3x^2 - 18$. Is $k = 2$ a zero?

Use synthetic division with $k = 2$.

2)	2	−3	0	−18	
		4	2	4	
	2	1	2	−14	← Remainder

Since the remainder is <u>not</u> 0, 2 is <u>not</u> a zero.

3.2 Example 3(b) Deciding Whether a Number is a Zero (page 325)

Let $f(x) = x^4 - 4x^3 - 14x^2 + 36x + 45$. Is $k = -3$ a zero?

Use synthetic division with $k = -3$.

−3)	1	−4	−14	36	45	
		−3	21	−21	−45	
	1	−7	7	15	0	← Remainder

–3 **is** a zero.

3.2 Example 3(c) Deciding Whether a Number is a Zero (page 325)

Let $f(x) = x^4 - x^3 + 6x^2 + 14x - 20$. Is $k = 1 + 3i$ a zero?

Use synthetic division with $k = 1 + 3i$.

$1+3i$)	1	−1	6	14	−20
		$1+3i$	$-9+3i$	$-12-6i$	20
	1	$3i$	$-3+3i$	$2-6i$	0 ↑ Remainder

1 + 3*i* **is** a zero.

3.3 Zeros of Polynomial Functions

Factor Theorem ▪ Rational Zeros Theorem ▪ Number of Zeros ▪ Conjugate Zeros Theorem ▪ Zeros of a Polynomial Function ▪ Descartes' Rule of Signs

3.3 Example 1(a) Deciding Whether $x - k$ is a Factor of $f(x)$ (page 328)

Let $f(x) = 3x^4 - 48x^2 + 8x + 32$. Is $x + 4$ a factor?

By the factor theorem $x + 4$ is a factor of $f(x)$ if and only if $f(-4) = 0$.

−4)	3	0	−48	8	32	Insert 0 as the coefficient for the missing x^3-term.
		−12	48	0	−32	
	3	−12	0	8	0	← Remainder

Since $f(-4) = 0$, $x + 4$ **is** a factor of $f(x)$.

3.3 **Example 1(b)** Deciding Whether $x - k$ is a Factor of $f(x)$ (page 328)

Let $f(x) = x^5 + 6x^4 + 11x^3 + 12x^2 + 5x - 20$.
Is $x + 4$ a factor?

By the factor theorem $x + 4$ is a factor of $f(x)$ if and only if $f(-4) = 0$.

$$\begin{array}{r|rrrrrr} -4 & 1 & 6 & 11 & 12 & 5 & -20 \\ & & -4 & -8 & -12 & 0 & -20 \\ \hline & 1 & 2 & 3 & 0 & 5 & \underset{\uparrow\ \text{Remainder}}{-40} \end{array}$$

Since $f(-4) \neq 0$, $x + 4$ is **not** a factor of $f(x)$.

3.3 **Example 2** Factoring a Polynomial Given a Zero (page 329)

Factor $f(x) = 6x^3 - 37x^2 + 32x + 15$ into linear factors if 5 is a zero of f.

Since 5 is a zero of f, $x - 5$ is a factor. Divide $f(x)$ by $x - 5$.

$$\begin{array}{r|rrrr} 5 & 6 & -37 & 32 & 15 \\ & & 30 & -35 & -15 \\ \hline & 6 & -7 & -3 & 0 \end{array}$$

The quotient is $6x^2 - 7x - 3$.

$$f(x) = (x-5)\underbrace{\left(6x^2 - 7x - 3\right)}$$

$$f(x) = (x-5)(2x-3)(3x+1)$$

3.3 **Example 3(a)** Using the Rational Zeros Theorem (page 330)

$$f(x) = 8x^4 - 26x^3 - 27x^2 + 11x + 4$$

List all possible rational zeros.

p must be a factor of $a_0 = 4$: ±1, ±2, ±4

q must be a factor of $a_4 = 8$: ±1, ±2, ±4, ±8

Possible rational zeros, $\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm 2, \pm 4$

3.3 **Example 3(b)** Using the Rational Zeros Theorem (page 330)

Find all rational zeros and factor $f(x)$ into linear factors.

Use trial and error to find the first factor. Is 1 a factor?

$$\begin{array}{r|rrrrr} 1 & 8 & -26 & -27 & 11 & 4 \\ & & 8 & -18 & -45 & -34 \\ \hline & 8 & -18 & -45 & -34 & -30 \leftarrow \text{Remainder} \end{array}$$

1 is **not** a factor.

Is –1 a factor?

$$\begin{array}{r|rrrrr} -1 & 8 & -26 & -27 & 11 & 4 \\ & & -8 & 34 & -7 & -4 \\ \hline & \underbrace{8 \quad -34 \quad 7 \quad 4}_{\text{Quotient}} & & & & 0 \leftarrow \text{Remainder} \end{array}$$

–1 **is** a factor.

3.3 **Example 3(b)** Using the Rational Zeros Theorem (cont.)

$$f(x) = (x+1)\left(8x^3 - 34x^2 + 7x + 4\right)$$

Use the quotient polynomial $8x^3 - 34x^2 + 7x + 4$ to find the next factor. Is 4 a factor?

$$\begin{array}{r|rrrr} 4 & 8 & -34 & 7 & 4 \\ & & 32 & -8 & -4 \\ \hline & 8 & -2 & -1 & 0 \leftarrow \text{Remainder} \end{array}$$

4 **is** a factor.

$$f(x) = (x+1)(x-4)\left(8x^2 - 2x - 1\right)$$
$$= (x+1)(x-4)(2x-1)(4x+1)$$

The rational zeros are $-1, 4, \frac{1}{2}, -\frac{1}{4}$.

3.3 **Example 4(a)** Finding a Polynomial Function that Satisfies Given Conditions (Real Zeros) (page 332)

Find a function f defined by a polynomial of degree 3 that has zeros of –3, –2, and 5, and $f(-1) = 6$.

The three zeros give $x - (-3) = x + 3$, $x - (-2) = x + 2$, and $x - 5$ as factors of f.

$$f(x) = a(x+3)(x+2)(x-5)$$

Since $f(-1) = 6$, we can solve for a:

$$f(-1) = a(-1+3)(-1+2)(-1-5)$$
$$6 = -12a \Rightarrow a = -\tfrac{1}{2}$$

$$f(x) = -\tfrac{1}{2}(x+3)(x+2)(x-5) = -\tfrac{1}{2}x^3 + \tfrac{19}{2}x + 15$$

3.3 **Example 4(b)** Finding a Polynomial Function that Satisfies Given Conditions (Real Zeros) (page 332)

Find a function f defined by a polynomial of degree 3 that has zero 4 of multiplicity 3, and $f(2) = –24$.

The function f has form

$$f(x) = a(x-4)(x-4)(x-4) = a(x-4)^3$$

Since $f(2) = –24$, we can solve for a:

$$f(2) = a(2-4)^3$$
$$-24 = -8a \Rightarrow a = 3$$

$$f(x) = 3(x-4)^3 = 3x^3 - 36x^2 + 144x - 192$$

3.3 **Example 5** Finding a Polynomial Function that Satisfies Given Conditions (Complex Zeros) (page 334)

Find a polynomial function of least degree having only real coefficients and zeros –4 and 3 – *i*.

By the conjugate zeros theorem, 3 + *i* is also a zero.

For the polynomial to be of least degree, –4, 3 – *i*, and 3 + *i* must be the only zeros.

3.3 **Example 5** Finding a Polynomial Function that Satisfies Given Conditions (Complex Zeros) (cont.)

$$\begin{aligned} f(x) &= [x-(-4)][x-(3-i)][x-(3+i)] \\ &= (x+4)[(x-3)-i][(x-3)+i] \\ &= (x+4)\left[(x-3)^2 - i^2\right] \\ &= (x+4)(x^2-6x+9+1) \\ &= (x+4)(x^2-6x+10) \\ &= x^3 - 2x^2 - 14x + 40 \end{aligned}$$

3.3 **Example 6** Finding All Zeros of a Polynomial Function Given One Zero (page 335)

Find all zeros of $f(x) = x^4 - x^3 - 17x^2 + 55x - 50$ given that 2 + *i* is a zero.

Since $f(x)$ has only real coefficients, and 2 + *i* is a zero, then 2 – *i* is also a zero.

Use synthetic division to divide $f(x)$ by $x – (2 + i)$.

$$\begin{array}{r|rrrrr} 2+i & 1 & -1 & -17 & 55 & -50 \\ & & 2+i & 1+3i & -35-10i & 50 \\ \hline & 1 & 1+i & -16+3i & 20-10i & 0 \end{array}$$

3.3 **Example 6** Finding All Zeros of a Polynomial Function Given One Zero (cont.)

Now use synthetic division to divide the quotient polynomial $x^3 + (1+i)x^2 + (-16+3i)x + (20-10i)$ by $x – (2 – i)$.

$$\begin{array}{r|rrrr} 2-i & 1 & 1+i & -16+3i & 20-10i \\ & & 2-i & 6-3i & -20+10i \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

$$\begin{aligned} f(x) &= [x-(2+i)][x-(2-i)](x^2+3x-10) \\ &= [x-(2+i)][x-(2-i)](x+5)(x-2) \end{aligned}$$

The zeros of $f(x)$ are 2 + *i*, 2 – *i*, –5, and 2.

3.3 **Example 7** Applying Descartes' Rule of Signs (page 336)

Determine the possible number of positive real zeros and negative real zeros of

$$f(x) = -2x^4 + 3x^3 - 5x^2 + 4x - 1.$$

$f(x)$ has 4 variations in sign:

$$f(x) = \underbrace{-2x^4 +}_{1}\underbrace{3x^3 -}_{2}\underbrace{5x^2 +}_{3}\underbrace{4x -}_{4}1$$

Thus, f has 4, or 4 – 2 = 2, or 2 – 2 = 0 positive real zeros.

3.3 **Example 7** Applying Descartes' Rule of Signs (cont.)

For negative zeros, consider the variations in signs for *f*(–*x*).

$$f(-x) = -2(-x)^4 + 3(-x)^3 - 5(-x)^2 + 4(-x) - 1$$
$$= -2x^4 - 3x^3 - 5x^2 - 4x - 1$$

There are no changes in sign, so there are no negative zeros.

3.4 Polynomial Functions: Graphs, Applications, and Models

Graphs of $f(x) = ax^n$ ▪ Graphs of General Polynomial Functions ▪ Turning Points and End Behavior ▪ Graphing Techniques ▪ Intermediate Value and Boundedness Theorems ▪ Approximating Real Zeros ▪ Polynomial Models and Curve Fitting

3.4 **Example 1(a)** Graphing Functions of the Form $f(x) = ax^n$ ($a = 1$) (page 340)

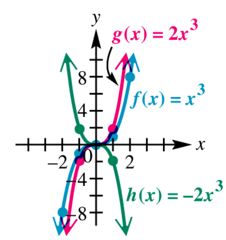
Graph $f(x) = x^3, g(x) = 2x^3$, and $h(x) = -2x^3$.

Choose several values for *x*, and find the corresponding values of *f*(*x*), *g*(*x*), and *h*(*x*).

x	$f(x) = x^3$	$g(x) = 2x^3$	$h(x) = -2x^3$
−2	−8	−16	16
−1	−1	−2	2
0	0	0	0
1	1	2	−2
2	8	16	−16

3.4 **Example 1(a)** Graphing Functions of the Form $f(x) = ax^n$ ($a = 1$) (cont.)

Plot the ordered pairs, and connect the points with a smooth curve.



3.4 **Example 1(b)** Graphing Functions of the Form $f(x) = ax^n$ ($a = 1$) (page 340)

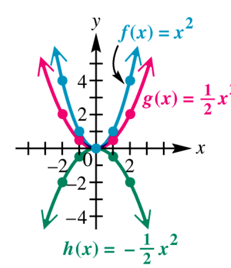
Graph $f(x) = x^2, g(x) = \frac{1}{2}x^2$, and $h(x) = -\frac{1}{2}x^2$.

Choose several values for *x*, and find the corresponding values of *f*(*x*), *g*(*x*), and *h*(*x*).

x	$f(x) = x^2$	$g(x) = \frac{1}{2}x^2$	$h(x) = -\frac{1}{2}x^2$
−2	4	2	−2
−1	1	$\frac{1}{2}$	$-\frac{1}{2}$
0	0	0	0
1	1	$\frac{1}{2}$	$-\frac{1}{2}$
2	4	2	−2

3.4 **Example 1(b)** Graphing Functions of the Form $f(x) = ax^n$ ($a = 1$) (cont.)

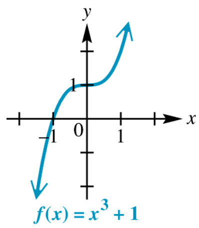
Plot the ordered pairs, and connect the points with a curve.



3.4 **Example 2(a)** Examining Vertical and Horizontal Translations (page 341)

Graph $f(x) = x^3 + 1$.

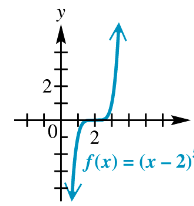
The graph of $f(x) = x^3 + 1$ is the same as the graph of $f(x) = x^3$, but translated 1 unit up. It includes the points (–1, 0), (0, 1), and (1, 2).



3.4 **Example 2(b)** Examining Vertical and Horizontal Translations (page 341)

Graph $f(x) = (x - 2)^5$.

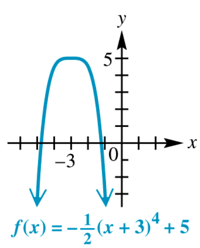
The graph of $f(x) = (x - 2)^5$ is the same as the graph of $f(x) = x^5$, but translated 2 units right. It includes the points (1, –1), (2, 0), and (3, 1).



3.4 **Example 2(c)** Examining Vertical and Horizontal Translations (page 341)

Graph $f(x) = -\frac{1}{2}(x + 3)^4 + 5$.

The graph of $f(x) = -\frac{1}{2}(x + 3)^4 + 5$ is the same as the graph of $f(x) = x^4$, but translated 3 units left, reflected across the x-axis, stretched vertically by a factor of ½, and then translated 5 units up. It includes the points (–2, 1.5), (–3, 2), and (–4, 1.5).



3.4 **Example 3** Determining End Behavior Given The Defining Polynomial (page 344)

Use the symbols for end behavior to describe the end behavior of the graph of each function.

(a) $f(x) = -x^4 + 2x^2 + x - 8$

Since $a = -1 < 0$ and f has even degree, the end behavior is shaped

(b) $g(x) = x^3 + 3x^2 - x + 5$

Since $a = 1 > 0$ and g has odd degree, the end behavior is shaped

3.4 **Example 3** Determining End Behavior Given The Defining Polynomial (cont.)

(c) $h(x) = -x^5 + x^4 - 3x^2 + 2$

Since $a = -1 < 0$ and h has odd degree, the end behavior is shaped

(d) $k(x) = x^6 + x^3 + 1$

Since $a = 1 > 0$ and f has even degree, the end behavior is shaped

3.4 **Example 4** Graphing a Polynomial Function (page 345)

Graph $f(x) = 2x^3 + 3x^2 - 11x - 6$.

Step 1: p must be a factor of $a_0 = -6$ and q must be a factor of $a_3 = 2$. Thus, p can be ±1, ±2, ±3, ±6, and q can be ±1 or ±2. The possible rational zeros, $\frac{p}{q}$, are ±1, ±2, ±3, ±6, $\pm\frac{1}{2}$ or $\pm\frac{3}{2}$. The remainder theorem shows that 2 is a zero.

$$\begin{array}{r|rrrr} 2 & 2 & 3 & -11 & -6 \\ & & 4 & 14 & 6 \\ \hline & 2 & 7 & 3 & 0 \end{array}$$

3.4 **Example 4** Graphing a Polynomial Function (cont.)

$$f(x) = 2x^3 + 3x^2 - 11x - 6 = (x-2)(2x^2 + 7x + 3)$$
$$= (x-2)(2x+1)(x+3)$$

Setting each factor equal to zero gives the zeros of f as 2, $-\frac{1}{2}$, and –3.

Step 2: $f(0) = -6$, so plot $(0, -6)$.

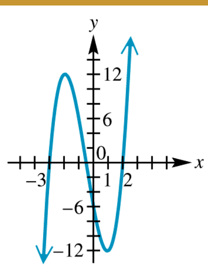
Step 3: The x-intercepts divide the x-axis into four intervals. Select an x-value in each interval and substitute it into the equation to determine whether the values of the function are positive or negative in that interval.

3.4 **Example 4** Graphing a Polynomial Function (cont.)

Interval	Test Point	Value of $f(x)$	Sign of $f(x)$	Graph Above or Below x-Axis
$(-\infty, -3)$	–5	–126	Negative	Below
$\left(-3, -\frac{1}{2}\right)$	–2	12	Positive	Above
$\left(-\frac{1}{2}, 2\right)$	1	–12	Negative	Below
$(2, \infty)$	3	42	Positive	Above

Plot the x-intercepts, y-intercept, and test points with a smooth curve to obtain the graph.

3.4 **Example 4** Graphing a Polynomial Function (cont.)



$f(x) = 2x^3 + 3x^2 - 11x - 6$
$f(x) = (x+3)(2x+1)(x-2)$

3.4 **Example 5** Locating a Zero (page 346)

Use synthetic division and a graph to show that $f(x) = -x^3 + 2x^2 - 4x + 5$ has a real zero between 1 and 2.

Use synthetic division to find $f(1)$ and $f(2)$.

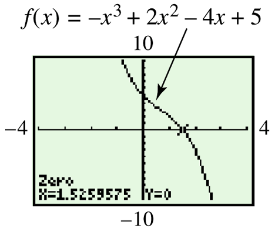
$$\begin{array}{r|rrrr} 1 & -1 & 2 & -4 & 5 \\ & & -1 & 1 & -3 \\ \hline & -1 & 1 & -3 & 2 \end{array} \qquad \begin{array}{r|rrrr} 2 & -1 & 2 & -4 & 5 \\ & & -2 & 0 & -8 \\ \hline & -1 & 0 & -4 & -3 \end{array}$$

$f(1) = 2$ $\qquad$ $f(2) = -3$

By the Intermediate Value Theorem, there is a real zero between 1 and 2.

3.4 **Example 5** Locating a Zero (cont.)

$f(x) = -x^3 + 2x^2 - 4x + 5$



3.4 **Example 6(a)** Using the Boundedness Theorem (page 348)

Show that $f(x) = x^4 + 5x^2 + 3x - 7$ has no real zero greater than 1.

$f(x)$ has real coefficients and the leading coefficient, 1, is positive, so the Boundedness Theorem applies. Divide $f(x)$ synthetically by $x - 1$.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 5 & 3 & -7 \\ & & 1 & 1 & 6 & 9 \\ \hline & 1 & 1 & 6 & 9 & 2 \end{array}$$

Since $1 > 0$ and all the numbers in the last row of the synthetic division are nonnegative, $f(x)$ has no real zeros greater than 1.

3.4 **Example 6(b)** Using the Boundedness Theorem (page 348)

Show that $f(x) = x^4 + 5x^2 + 3x - 7$ has no real zero less than –2.

$f(x)$ has real coefficients and the leading coefficient, 1, is positive, so the Boundedness Theorem applies. Divide $f(x)$ synthetically by $x + 2$.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & 5 & 3 & -7 \\ & & -2 & 4 & -18 & 30 \\ \hline & 1 & -2 & 9 & -15 & 23 \end{array}$$

Since –2 < 0 and the numbers in the last row of the synthetic division alternate in sign, $f(x)$ has no real zeros less than –2.

3.4 **Example 7** Approximating Real Zeros of a Polynomial Function (page 349)

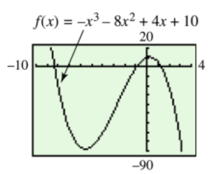
Approximate the real zeros of $f(x) = -x^3 - 8x^2 + 4x + 10.$

The greatest degree term is x^3, so the graph will have end behavior similar to $f(x) = -x^3$.

There are at most 3 real zeros.

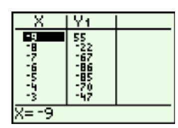
$f(0) = 10$, so the y-intercept is 10.

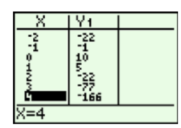
3.4 **Example 7** Approximating Real Zeros of a Polynomial Function (cont.)



The graph shows that there are 3 real zeros.

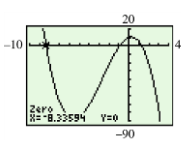
3.4 **Example 7** Approximating Real Zeros of a Polynomial Function (cont.)

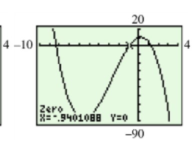


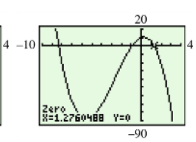


There are sign changes between –9 and –8, –1 and 0, and 1 and 2, thus the zeros are between –9 and –8, –1 and 0, and 1 and 2.

3.4 **Example 7** Approximating Real Zeros of a Polynomial Function (cont.)







The zeros are approximately –8.33594, –.9401088, and 1.2760488.

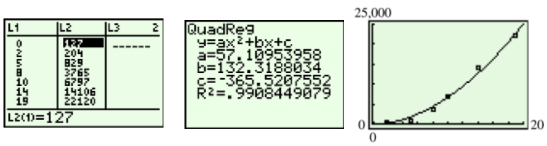
3.4 **Example 8(a)** Examining a Polynomial Model for Debit Card Use (page 350)

The table shows the number of transactions, in millions, by users of bank debit cards.

Year	Transactions (in millions)
1990	127
1992	204
1995	829
1998	3765
2000	6797
2004	14,106
2009	22,120

Using the data in the table, with $x = 0$ representing 1990, $x = 5$ representing 1995, etc., use the regression feature of a calculator to determine the quadratic function that best fits the data. Plot the data and the graph.

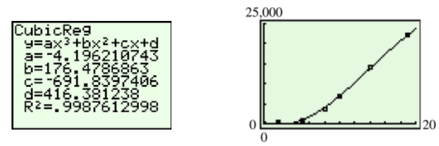
3.4 **Example 8(a)** Examining a Polynomial Model for Debit Card Use (page 350)



The best fitting quadratic function for the data is
$y = 57.11x^2 + 132.3x - 365.5$

3.4 **Example 8(b)** Examining a Polynomial Model for Debit Card Use (page 350)

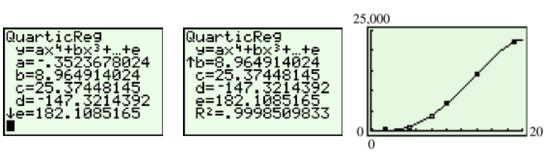
Repeat part (a) for a cubic function.



The best fitting cubic function for the data is
$y = -4.196x^3 + 176.5x^2 - 691.8x + 416.4$

3.4 **Example 8(c)** Examining a Polynomial Model for Debit Card Use (page 350)

Repeat part (a) for a quartic function.



The best fitting quartic function for the data is
$y = -.3524x^4 + 8.965x^3 + 25.37x^2 - 147.3x + 182.1$

3.4 **Example 8(d)** Examining a Polynomial Model for Debit Card Use (page 350)

Compare R^2, the square of the correlation coefficient, for the three functions to decide which function best fits the data.

Quadratic: $R^2 = .9908449079$

Cubic: $R^2 = .9987612998$

Quartic: $R^2 = .9998509833$

The quartic value is closest to 1, so the quartic function is the best fit.