95. *Height of a Projectile* A projectile is fired straight up from ground level. After t seconds, its height above the ground is s feet, where

$$s = -16t^2 + 220t.$$

For what time period is the projectile at least 624 ft above the ground?

96. *Velocity of an Object* Suppose the velocity of an object is given by

$$v = 2t^2 - 5t - 12,$$

where t is time in seconds. (Here t can be positive or negative.) Find the intervals where the velocity is negative.

97. *Cancer Risk from Radon Gas Exposure* Radon gas occurs naturally in homes and is produced when uranium radioactively decays into lead. Exposure to radon gas is a known lung cancer risk. According to the Environmental Protection Agency (EPA) the individual lifetime excess cancer risk R for radon exposure is between

$$1.5 \times 10^{-3} \quad \text{and} \quad 6.0 \times 10^{-3},$$

where $R = .01$ represents a 1% increase in risk of developing lung cancer. (*Source: Indoor-Air Assessment: A Review of Indoor Air Quality Risk Characterization Studies.* Report No. EPA/600/8-90/044, Environmental Protection Agency, 1991.)

(a) Calculate the range of individual annual risk by dividing R by an average life expectancy of 72 yr.

(b) Approximate the range of new cases of lung cancer each year (to the nearest hundred) caused by radon if the population of the United States is 310 million.

98. A student attempted to solve the inequality

$$\frac{2x-3}{x+2} \le 0$$

by multiplying both sides by $x + 2$ to get

$$2x - 3 \le 0$$

$$x \le \frac{3}{2}.$$

He wrote the solution set as $\left(-\infty, \frac{3}{2}\right]$. Is his solution correct? Explain.

99. A student solved the inequality $x^2 \le 144$ by taking the square root of both sides to get $x \le 12$. She wrote the solution set as $(-\infty, 12]$. Is her solution correct? Explain.

100. *Concept Check* Use the discriminant to find the values of k for which $x^2 - kx + 8 = 0$ has two real solutions.

1.8 Absolute Value Equations and Inequalities

Absolute Value Equations ▪ Absolute Value Inequalities ▪ Special Cases ▪ Absolute Value Models for Distance and Tolerance

Recall from **Section R.2** that the **absolute value** of a number a, written $|a|$, gives the distance from a to 0 on a number line. By this definition, the equation $|x| = 3$ can be solved by finding all real numbers at a distance of 3 units from 0. As shown in Figure 19 on the next page, two numbers satisfy this equation, 3 and -3, so the solution set is $\{-3, 3\}$.

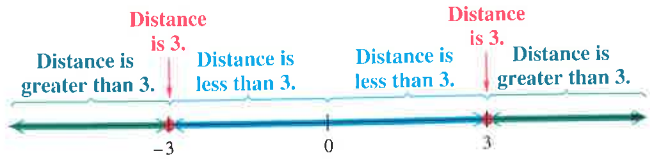


Figure 19

Similarly, $|x| < 3$ is satisfied by all real numbers whose distances from 0 are less than 3, that is, the interval $-3 < x < 3$ or $(-3, 3)$. See Figure 19. Finally, $|x| > 3$ is satisfied by all real numbers whose distances from 0 are greater than 3. As Figure 19 shows, these numbers are less than -3 or greater than 3, so the solution set is $(-\infty, -3) \cup (3, \infty)$. Notice in Figure 19 that the union of the solution sets of $|x| = 3$, $|x| < 3$, and $|x| > 3$ is the set of real numbers.

These observations support the following properties of absolute value.

PROPERTIES OF ABSOLUTE VALUE

1. **For $b > 0$, $|a| = b$ if and only if $a = b$ or $a = -b$.**
2. **$|a| = |b|$ if and only if $a = b$ or $a = -b$.**

For any positive number b:

3. **$|a| < b$ if and only if $-b < a < b$.**
4. **$|a| > b$ if and only if $a < -b$ or $a > b$.**

Absolute Value Equations

We use Properties 1 and 2 to solve absolute value equations.

▶ EXAMPLE 1 SOLVING ABSOLUTE VALUE EQUATIONS

Solve each equation.

(a) $|5 - 3x| = 12$ **(b)** $|4x - 3| = |x + 6|$

Solution

(a) For the given expression $5 - 3x$ to have absolute value 12, it must represent either 12 or -12. This requires applying Property 1, with $a = 5 - 3x$ and $b = 12$.

$$|5 - 3x| = 12$$

Don't forget this second case.

$$5 - 3x = 12 \quad \text{or} \quad 5 - 3x = -12 \qquad \text{Property 1}$$

$$-3x = 7 \quad \text{or} \quad -3x = -17 \qquad \text{Subtract 5.}$$

$$x = -\frac{7}{3} \quad \text{or} \quad x = \frac{17}{3} \qquad \text{Divide by } -3.$$

Check the solutions $-\frac{7}{3}$ and $\frac{17}{3}$ by substituting them in the original absolute value equation. The solution set is $\left\{-\frac{7}{3}, \frac{17}{3}\right\}$.

(b)

$$|4x - 3| = |x + 6|$$

$$4x - 3 = x + 6 \quad \text{or} \quad 4x - 3 = -(x + 6) \qquad \text{Property 2}$$

$$3x = 9 \quad \text{or} \quad 4x - 3 = -x - 6$$

$$x = 3 \quad \text{or} \quad 5x = -3$$

$$x = -\frac{3}{5}$$

Check: $|4x - 3| = |x + 6|$ Original equation

If $x = -\frac{3}{5}$, then

$$\left|4\left(-\tfrac{3}{5}\right) - 3\right| = \left|-\tfrac{3}{5} + 6\right| \quad ?$$

$$\left|-\tfrac{12}{5} - 3\right| = \left|-\tfrac{3}{5} + 6\right| \quad ?$$

$$\left|-\tfrac{27}{5}\right| = \left|\tfrac{27}{5}\right|. \qquad \text{True}$$

If $x = 3$, then

$$|4(3) - 3| = |3 + 6| \quad ?$$

$$|12 - 3| = |3 + 6| \quad ?$$

$$|9| = |9|. \qquad \text{True}$$

Both solutions check. The solution set is $\left\{-\frac{3}{5}, 3\right\}$.

NOW TRY EXERCISES 9 AND 19. ◀

Absolute Value Inequalities

We use Properties 3 and 4 to solve absolute value inequalities.

▶ EXAMPLE 2 SOLVING ABSOLUTE VALUE INEQUALITIES

Solve each inequality.

(a) $|2x + 1| < 7$ **(b)** $|2x + 1| > 7$

Solution

(a) Use Property 3, replacing a with $2x + 1$ and b with 7.

$$|2x + 1| < 7$$

$$-7 < 2x + 1 < 7 \qquad \text{Property 3}$$

$$-8 < 2x < 6 \qquad \text{Subtract 1 from each part. (Section 1.7)}$$

$$-4 < x < 3 \qquad \text{Divide each part by 2.}$$

The final inequality gives the solution set $(-4, 3)$.

(b)

$$|2x + 1| > 7$$

$$2x + 1 < -7 \quad \text{or} \quad 2x + 1 > 7 \qquad \text{Property 4}$$

$$2x < -8 \quad \text{or} \quad 2x > 6 \qquad \text{Subtract 1 from each side.}$$

$$x < -4 \quad \text{or} \quad x > 3 \qquad \text{Divide each side by 2.}$$

The solution set is $(-\infty, -4) \cup (3, \infty)$.

NOW TRY EXERCISES 27 AND 29. ◀

Properties 1, 3, and 4 require that the absolute value expression be alone on one side of the equation or inequality.

▶ EXAMPLE 3 SOLVING AN ABSOLUTE VALUE INEQUALITY REQUIRING A TRANSFORMATION

Solve $|2 - 7x| - 1 > 4$.

Solution

$$|2 - 7x| - 1 > 4$$

$$|2 - 7x| > 5 \qquad \text{Add 1 to each side.}$$

$$2 - 7x < -5 \quad \text{or} \quad 2 - 7x > 5 \qquad \text{Property 4}$$

$$-7x < -7 \quad \text{or} \quad -7x > 3 \qquad \text{Subtract 2.}$$

$$x > 1 \quad \text{or} \quad x < -\frac{3}{7} \qquad \text{Divide by } -7\text{; reverse the direction of each inequality. (Section 1.7)}$$

The solution set is $\left(-\infty, -\frac{3}{7}\right) \cup (1, \infty)$.

NOW TRY EXERCISE 47. ◀

Special Cases Three of the four properties given in this section require the constant b to be positive. ***When $b \leq 0$, use the fact that the absolute value of any expression must be nonnegative and consider the truth of the statement.***

▶ EXAMPLE 4 SOLVING SPECIAL CASES OF ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Solve each equation or inequality.

(a) $|2 - 5x| \geq -4$ **(b)** $|4x - 7| < -3$ **(c)** $|5x + 15| = 0$

Solution

(a) Since the absolute value of a number is always nonnegative, the inequality $|2 - 5x| \geq -4$ is always true. The solution set includes all real numbers, written $(-\infty, \infty)$.

(b) There is no number whose absolute value is less than -3 (or less than *any* negative number). The solution set of $|4x - 7| < -3$ is $\emptyset$.

(c) The absolute value of a number will be 0 only if that number is 0. Therefore, $|5x + 15| = 0$ is equivalent to

$$5x + 15 = 0,$$

which has solution set $\{-3\}$. Check by substituting into the original equation.

NOW TRY EXERCISES 55, 57, AND 59. ◀

Absolute Value Models for Distance and Tolerance Recall from **Section R.2** that if a and b represent two real numbers, then the absolute value of their difference, either $|a - b|$ or $|b - a|$, represents the distance between them. This fact is used to write absolute value equations or inequalities to express distances.

▶ EXAMPLE 5 USING ABSOLUTE VALUE INEQUALITIES TO DESCRIBE DISTANCES

Write each statement using an absolute value inequality.

(a) k is no less than 5 units from 8. **(b)** n is within .001 unit of 6.

Solution

(a) Since the distance from k to 8, written $|k - 8|$ or $|8 - k|$, is no less than 5, the distance is greater than or equal to 5. This can be written as

$$|k - 8| \geq 5, \quad \text{or equivalently} \quad |8 - k| \geq 5.$$

Either form is acceptable.

(b) This statement indicates that the distance between n and 6 is less than .001, written

$$|n - 6| < .001, \quad \text{or equivalently} \quad |6 - n| < .001.$$

NOW TRY EXERCISES 81 AND 83. ◀

In quality control and other applications, as well as in more advanced mathematics, we often wish to keep the difference between two quantities within some predetermined amount, called the **tolerance.**

▶ EXAMPLE 6 USING ABSOLUTE VALUE TO MODEL TOLERANCE

Suppose $y = 2x + 1$ and we want y to be within .01 unit of 4. For what values of x will this be true?

Solution

$$|y - 4| < .01 \quad \text{Write an absolute value inequality.}$$
$$|2x + 1 - 4| < .01 \quad \text{Substitute } 2x + 1 \text{ for } y.$$
$$|2x - 3| < .01$$
$$-.01 < 2x - 3 < .01 \quad \text{Property 3}$$
$$2.99 < 2x < 3.01 \quad \text{Add 3 to each part.}$$
$$1.495 < x < 1.505 \quad \text{Divide each part by 2.}$$

Reversing these steps shows that keeping x in the interval $(1.495, 1.505)$ ensures that the difference between y and 4 is within .01 unit.

NOW TRY EXERCISE 87. ◀

LOOKING AHEAD TO CALCULUS

The precise definition of a **limit** in calculus requires writing absolute value inequalities as in Examples 5 and 6.

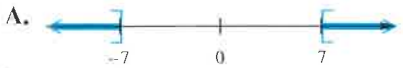
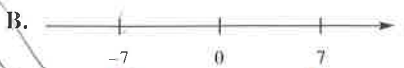
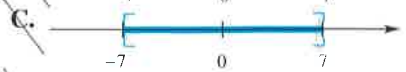
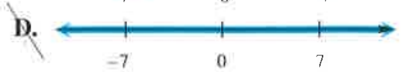
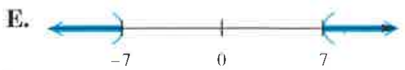
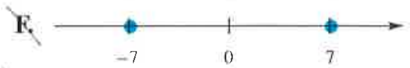
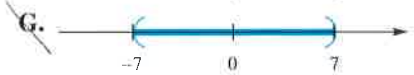
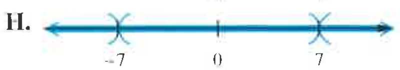
A standard problem in calculus is to find the "interval of convergence" of something called a **power series**, by solving an inequality of the form

$$|x - a| < r.$$

This inequality says that x can be any number within r units of a on the number line, so its solution set is indeed an interval—namely the interval $(a - r, a + r)$.

1.8 Exercises

Concept Check *Match each equation or inequality in Column I with the graph of its solution set in Column II.*

I	II
1. $\lvert x \rvert = 7$	**A.** 
2. $\lvert x \rvert = -7$	**B.** 
3. $\lvert x \rvert > -7$	**C.** 
4. $\lvert x \rvert > 7$	**D.** 
5. $\lvert x \rvert < 7$	**E.** 
6. $\lvert x \rvert \geq 7$	**F.** 
7. $\lvert x \rvert \leq 7$	**G.** 
8. $\lvert x \rvert \neq 7$	**H.** 

Solve each equation. See Example 1.

9. $\lvert 3x - 1 \rvert = 2$ **10.** $\lvert 4x + 2 \rvert = 5$ **11.** $\lvert 5 - 3x \rvert = 3$

12. $\lvert 7 - 3x \rvert = 3$ **13.** $\left\lvert \frac{x - 4}{2} \right\rvert = 5$ **14.** $\left\lvert \frac{x + 2}{2} \right\rvert = 7$

15. $\left\lvert \frac{5}{x - 3} \right\rvert = 10$ **16.** $\left\lvert \frac{3}{2x - 1} \right\rvert = 4$ **17.** $\left\lvert \frac{6x + 1}{x - 1} \right\rvert = 3$

18. $\left\lvert \frac{2x + 3}{3x - 4} \right\rvert = 1$ **19.** $\lvert 2x - 3 \rvert = \lvert 5x + 4 \rvert$ **20.** $\lvert x + 1 \rvert = \lvert 1 - 3x \rvert$

21. $\lvert 4 - 3x \rvert = \lvert 2 - 3x \rvert$ **22.** $\lvert 3 - 2x \rvert = \lvert 5 - 2x \rvert$ **23.** $\lvert 5x - 2 \rvert = \lvert 2 - 5x \rvert$

24. The equation $\lvert 5x - 6 \rvert = 3x$ cannot have a negative solution. Why?

25. The equation $\lvert 7x + 3 \rvert = -5x$ cannot have a positive solution. Why?

26. ***Concept Check*** Determine the solution set of each equation by inspection.
(a) $-\lvert x \rvert = \lvert x \rvert$ **(b)** $\lvert -x \rvert = \lvert x \rvert$ **(c)** $\lvert x^2 \rvert = \lvert x \rvert$ **(d)** $-\lvert x \rvert = 9$

Solve each inequality. Give the solution set using interval notation. See Example 2.

27. $\lvert 2x + 5 \rvert < 3$ **28.** $\lvert 3x - 4 \rvert < 2$ **29.** $\lvert 2x + 5 \rvert \geq 3$

30. $\lvert 3x - 4 \rvert \geq 2$ **31.** $\left\lvert \frac{1}{2} - x \right\rvert < 2$ **32.** $\left\lvert \frac{3}{5} + x \right\rvert < 1$

33. $4\lvert x - 3 \rvert > 12$ **34.** $5\lvert x + 1 \rvert > 10$ **35.** $\lvert 5 - 3x \rvert > 7$

36. $\lvert 7 - 3x \rvert > 4$ **37.** $\lvert 5 - 3x \rvert \leq 7$ **38.** $\lvert 7 - 3x \rvert \leq 4$

39. $\left\lvert \frac{2}{3}x + \frac{1}{2} \right\rvert \leq \frac{1}{6}$ **40.** $\left\lvert \frac{5}{3} - \frac{1}{2}x \right\rvert > \frac{2}{9}$ **41.** $\lvert .01x + 1 \rvert < .01$

42. Explain why the equation $|x| = |-x|$ has infinitely many solutions.

Solve each equation or inequality. See Examples 3 and 4.

43. $|4x + 3| - 2 = -1$
44. $|8 - 3x| - 3 = -2$
45. $|6 - 2x| + 1 = 3$
46. $|4 - 4x| + 2 = 4$
47. $|3x + 1| - 1 < 2$
48. $|5x + 2| - 2 < 3$
49. $\left|5x + \frac{1}{2}\right| - 2 < 5$
50. $\left|2x + \frac{1}{3}\right| + 1 < 4$
51. $|10 - 4x| + 1 \geq 5$
52. $|12 - 6x| + 3 \geq 9$
53. $|3x - 7| + 1 < -2$
54. $|-5x + 7| - 4 < -6$

Solve each equation or inequality. See Example 4.

55. $|10 - 4x| \geq -4$
56. $|12 - 9x| \geq -12$
57. $|6 - 3x| < -11$
58. $|18 - 3x| < -13$
59. $|8x + 5| = 0$
60. $|7 + 2x| = 0$
61. $|4.3x + 9.8| < 0$
62. $|1.5x - 14| < 0$
63. $|2x + 1| \leq 0$
64. $|3x + 2| \leq 0$
65. $|3x + 2| > 0$
66. $|4x + 3| > 0$

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 67–70)

To see how to solve an equation that involves the absolute value of a quadratic polynomial, such as $|x^2 - x| = 6$, ***work Exercises 67–70 in order.***

67. For $x^2 - x$ to have an absolute value equal to 6, what are the two possible values that it may be? (*Hint:* One is positive and the other is negative.)
68. Write an equation stating that $x^2 - x$ is equal to the positive value you found in Exercise 67, and solve it using factoring.
69. Write an equation stating that $x^2 - x$ is equal to the negative value you found in Exercise 67, and solve it using the quadratic formula. (*Hint:* The solutions are not real numbers.)
70. Give the complete solution set of $|x^2 - x| = 6$, using the results from Exercises 68 and 69.

Use the method described in Relating Concepts Exercises 67–70 if applicable to solve each equation or inequality.

71. $|4x^2 - 23x - 6| = 0$
72. $|6x^3 + 23x^2 + 7x| = 0$
73. $|x^2 + 1| - |2x| = 0$
74. $\left|\frac{x^2 + 2}{x}\right| - \frac{11}{3} = 0$
75. $|x^4 + 2x^2 + 1| < 0$
76. $|x^4 + 2x^2 + 1| \geq 0$
77. $\left|\frac{x - 4}{3x + 1}\right| \geq 0$
78. $\left|\frac{9 - x}{7 + 8x}\right| \geq 0$
79. *Concept Check* Write an equation involving absolute value that says the distance between p and q is 2 units.
80. *Concept Check* Write an equation involving absolute value that says the distance between r and s is 6 units.